## 2 Mountain Forced Flows

Well-known weather phenomena directly related to flow over orography include

- mountain waves
- lee waves and clouds
- rotors and rotor clouds
- severe downslope windstorms
- lee vortices
- lee cyclogenesis
- frontal distortion across mountains
- cold-air damming
- track deflection of midlatitude and tropical cyclones
- coastally trapped disturbances
- orographically induced rain and flash flooding
- orographically influenced storm tracks.

A majority of these phenomena are mesocale and are induced by stably stratified flow over orography.

## 2.1 Mountain Waves

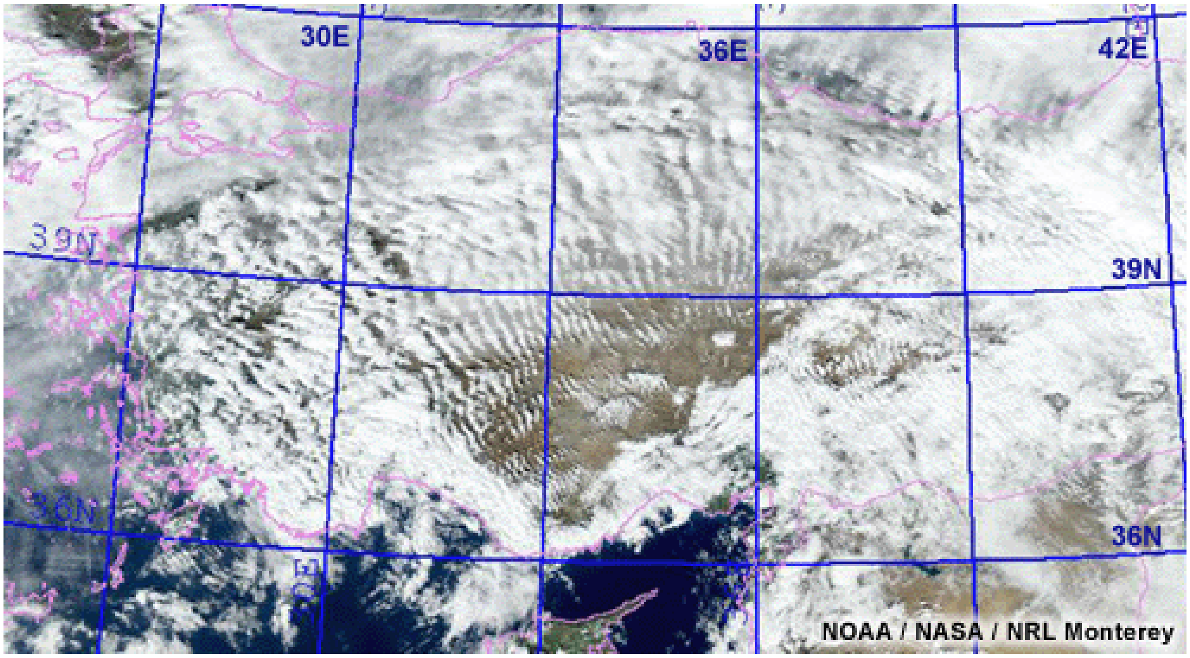


Figure 1: MODIS image of mountain wave clouds over Turkey. (credit: UCAR/COMET)

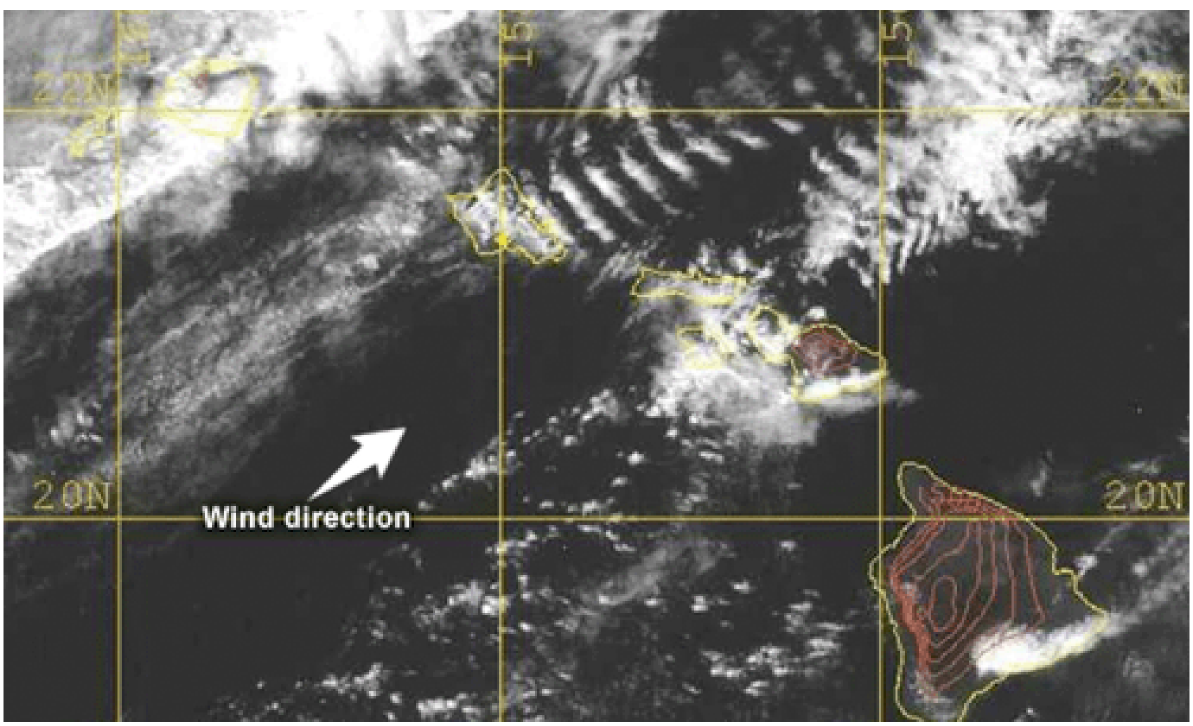


Figure 2: Trapped lee waves extend downwind from the Hawaiian Islands. (credit: UCAR/COMET)

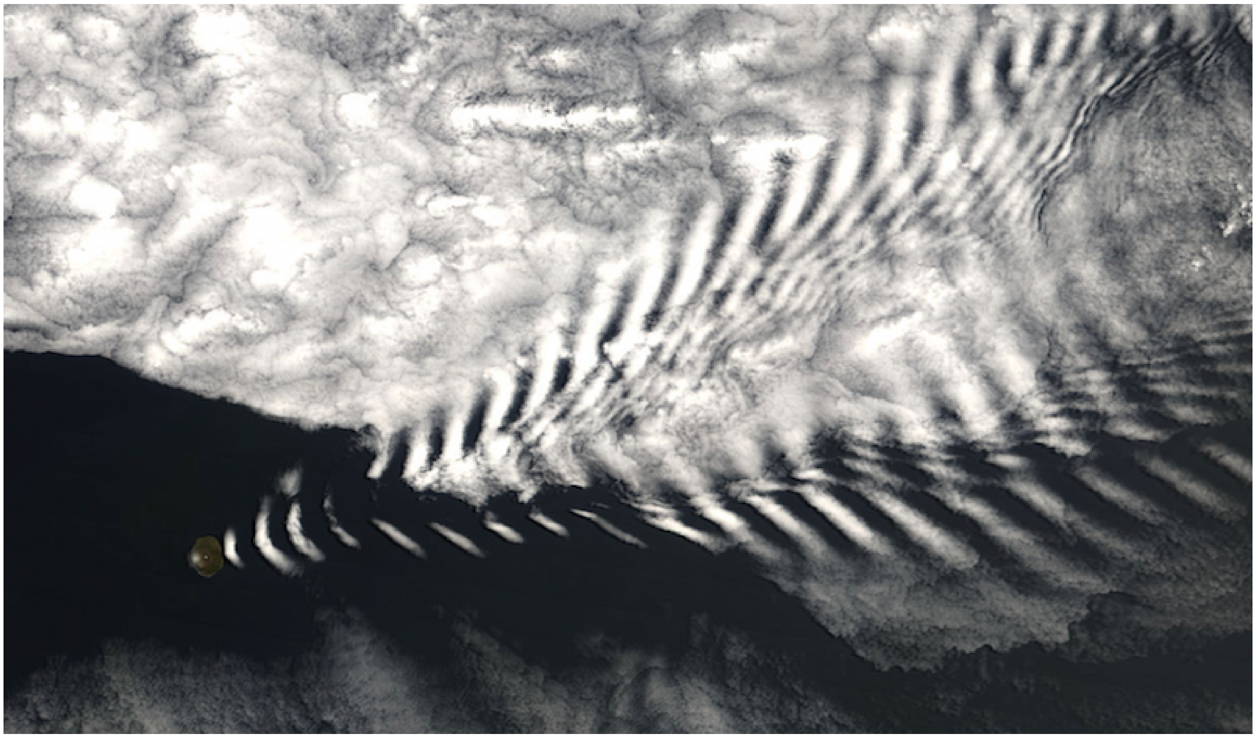

Figure 3: Wave clouds extend downwind from Amsterdam Island. (credit: Jeff Schmaltz/NASA)

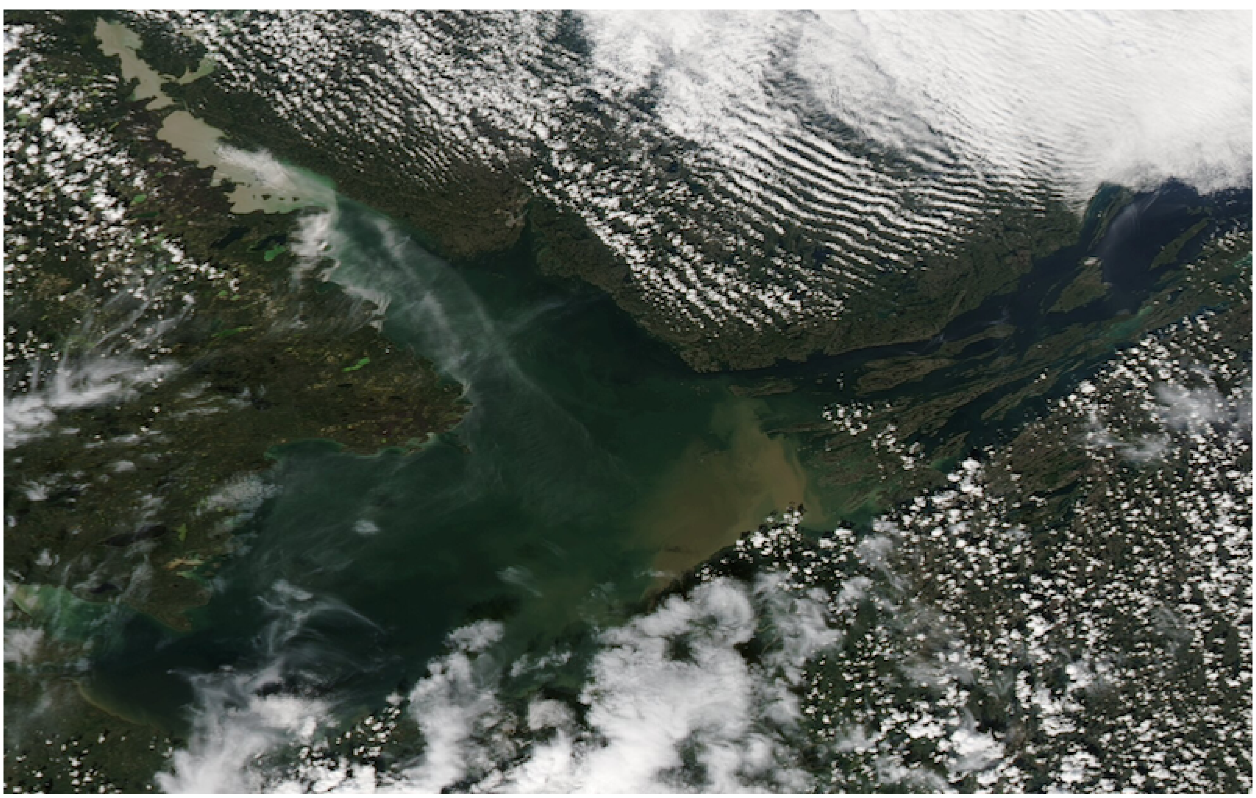

Figure 4: Wave clouds extend downwind over Canada's Great Slave Lake (credit: Earth Snapshot)

### 2.1.1 Primer: COMET Module

**Typical Mountain Waves Features**

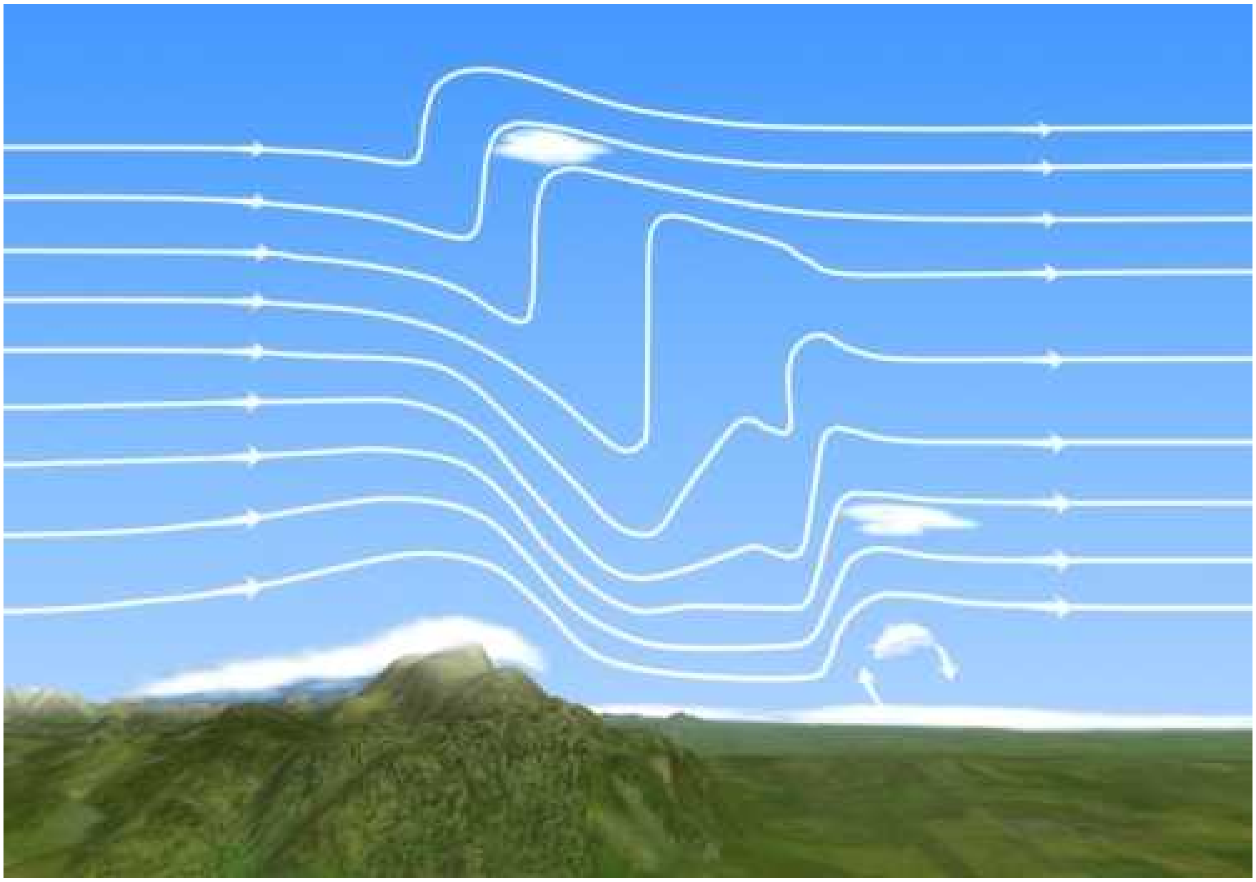

Figure 5: Typical features often associated with a mountain wave system. (credit: UCAR/COMET)

This figure shows the development of the typical features often associated with a mountain wave system. Notice the wind flow, with a strong component perpendicular to the primary ridge line. This is a typical condition for mountain wave development, as is a stable atmosphere. If air is being forced over the terrain, it will move downward along the lee slopes, then oscillate in a series of waves as it moves downstream. Sometimes these waves can propagate long distances in "lee wave trains."

**Cap Clouds**

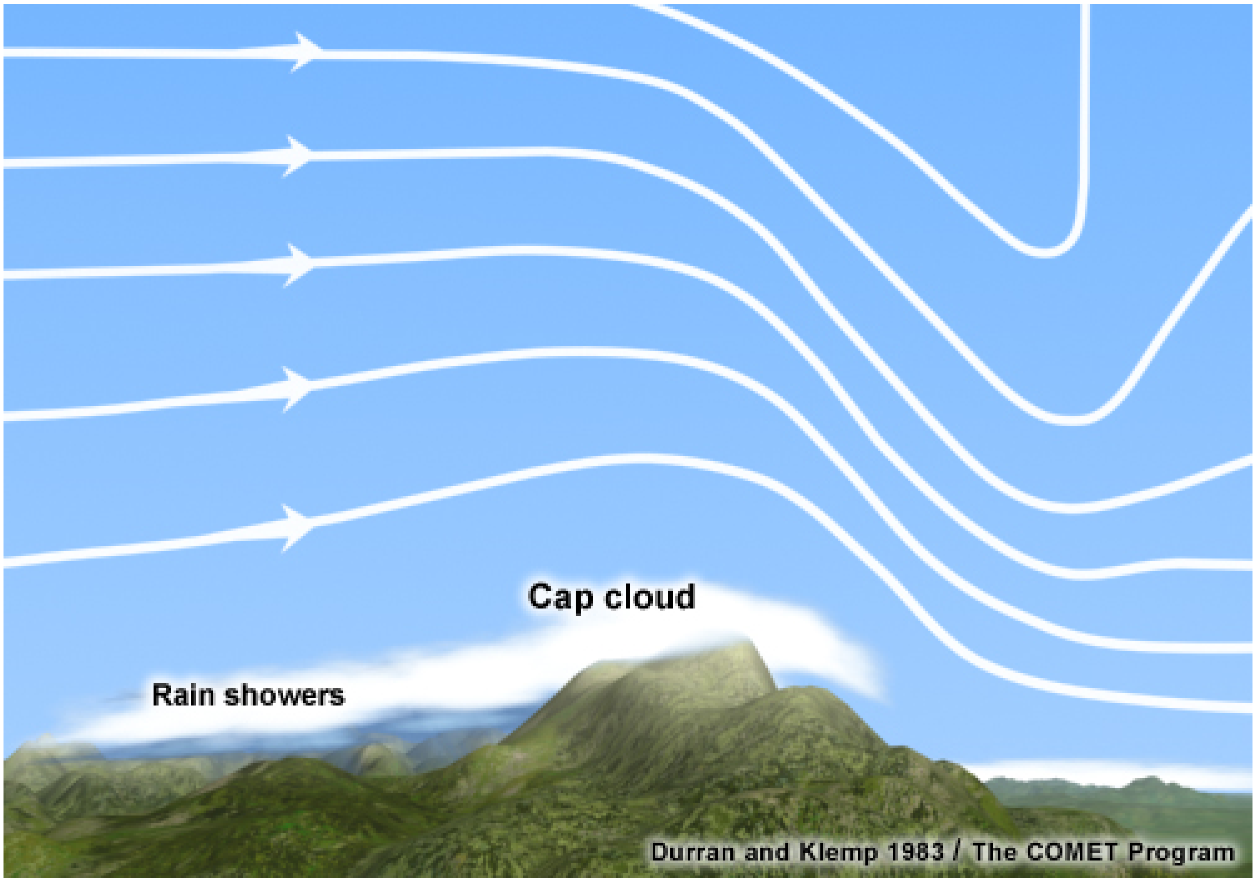


Figure 6: Depiction of a cap cloud. (credit: UCAR/COMET)

Cap clouds indicate likely wave activity downstream. They often appear along mountain ridges as air is forced up the windward side. If the flow is sufficiently humid, the moisture will condense into a cloud bank that follows mountain contours. Quite often, heavy orographic precipitation occurs on the upwind side of the barrier, particularly for barriers located near the sea. As the flow descends in the lee of the mountain ridge, the cloud evaporates. Viewed from downstream, cap clouds frequently appear as a wall of clouds hanging over the ridge top.

It is important to remember that while cap clouds indicate likely wave activity, their absence does not mean that waves are absent. Under drier conditions, waves may be present without cap clouds.

**The Vertically-Propagating Wave**

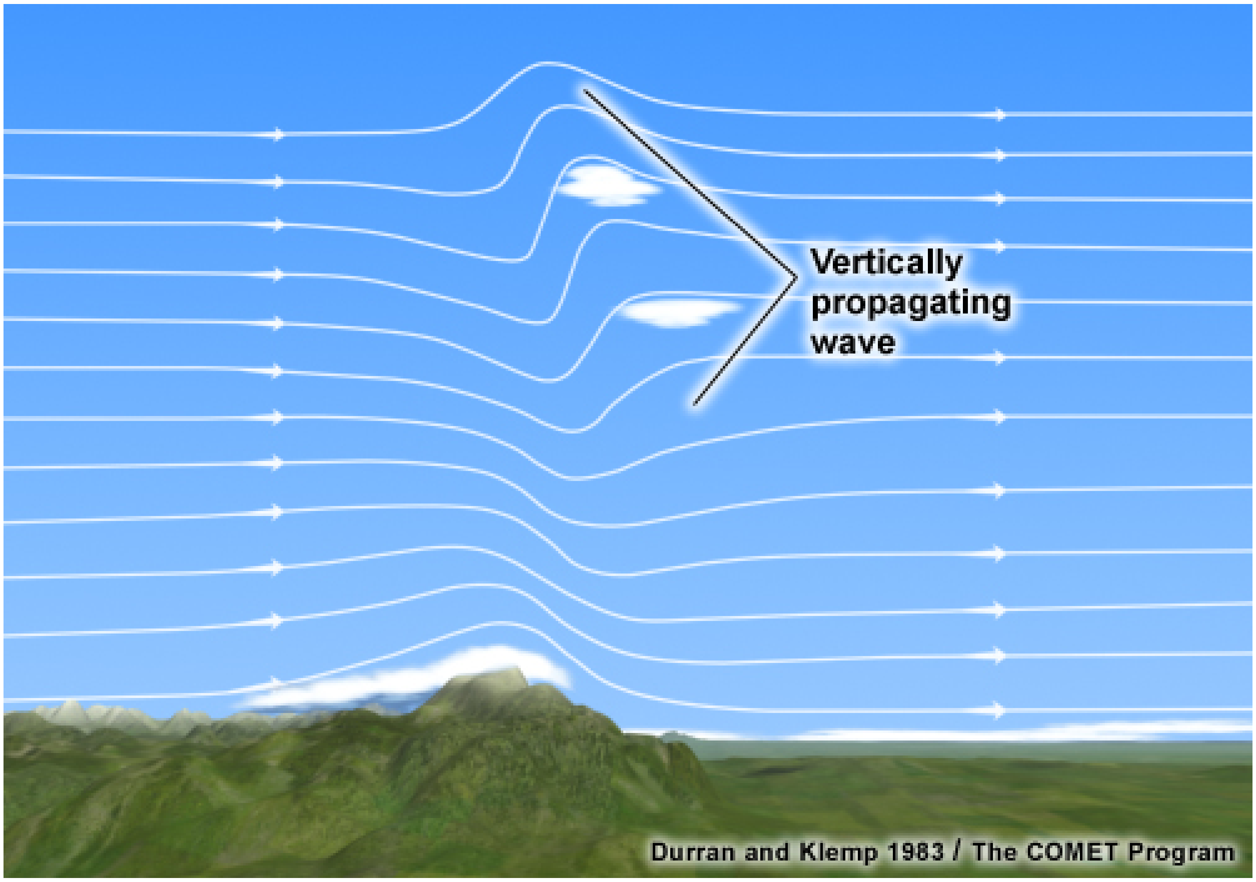


Figure 7: Depiction of a vertically-propagating wave. (credit: UCAR/COMET)

The vertically-propagating wave is often most severe within the first wavelength downwind of the mountain barrier. These waves frequently become more amplified and tilt upwind with height. Tilting, amplified waves can cause aircraft to experience turbulence at very high altitudes. Clear air turbulence often occurs near the tropopause due to vertically-propagating waves. Incredibly, these waves have been documented up to 70 $\mathrm{km}$ and higher.

**Breaking Waves**

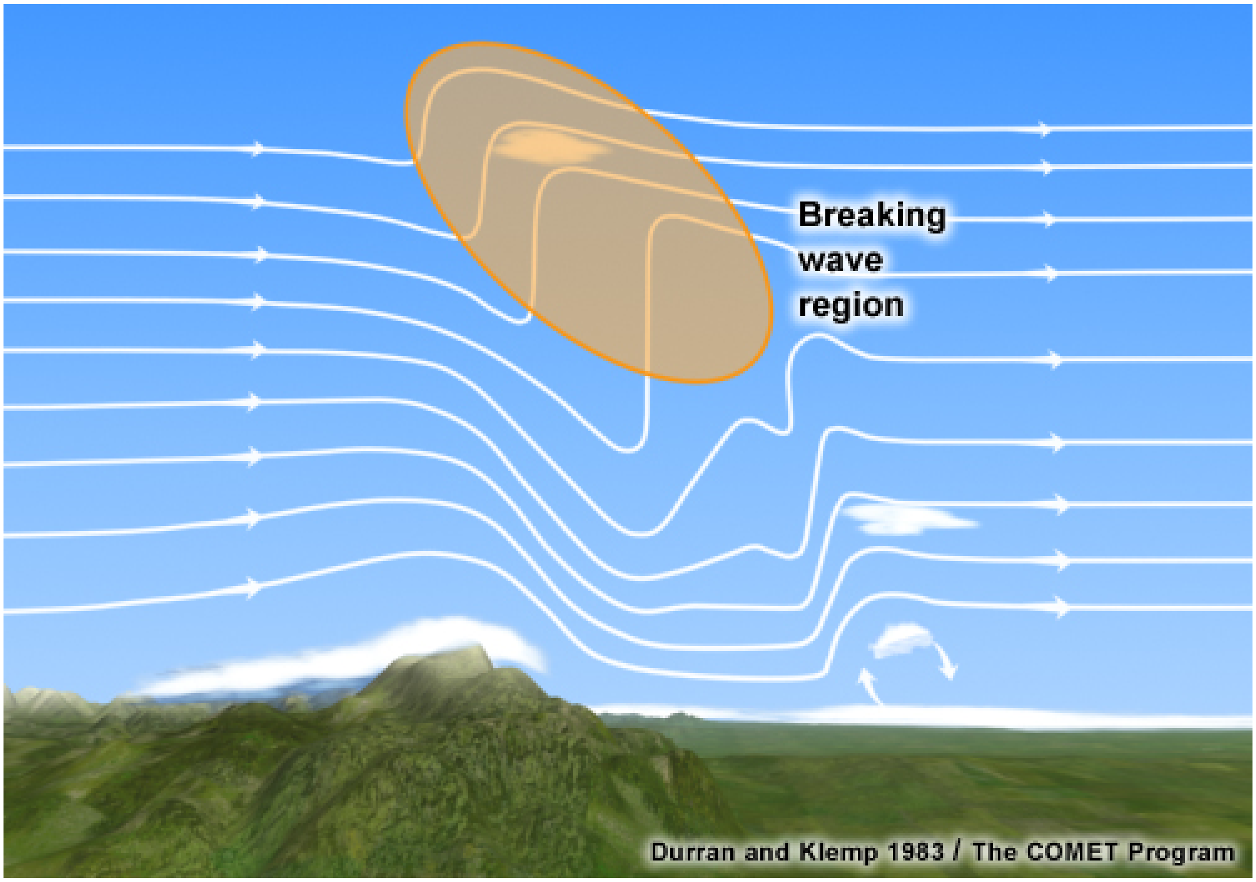


Figure 8: Depiction of a breaking wave. (credit: UCAR/COMET)

Vertically-propagating waves with sufficient amplitude may break in the troposphere or lower stratosphere. Wave-breaking can result in severe to extreme turbulence within the wave-breaking region and nearby, typically between 7 and $14\ \mathrm{km}$. If a vertically-propagating wave doesn't break, an aircraft would likely experience considerable wave action, but little turbulence.

**Downslope Winds**

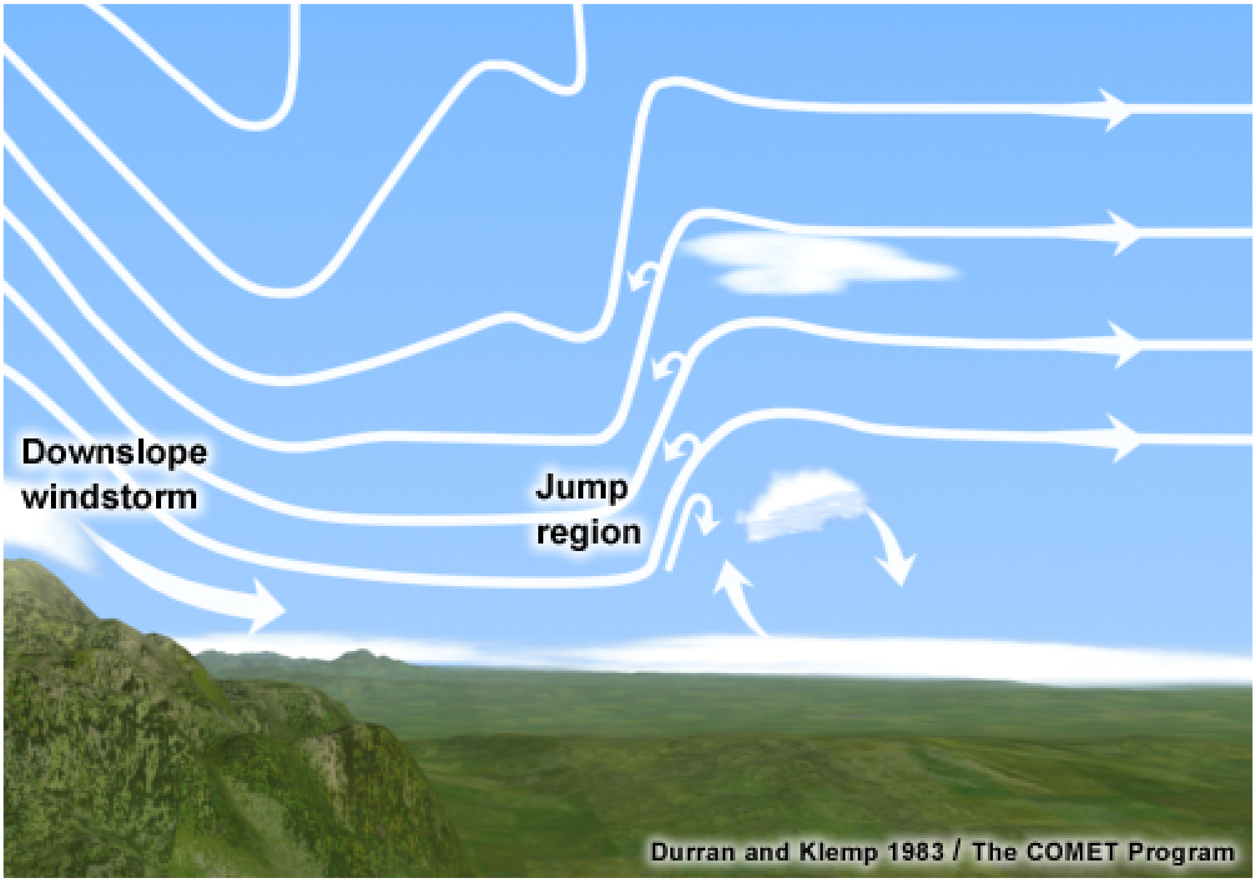


Figure 9: Depiction of a breaking wave. (credit: UCAR/COMET)

At times, strong downslope winds accompany mountain wave systems. Strong downslope wind cases are usually associated with strong cross-barrier flow, waves breaking aloft, and an inversion near the barrier top. In extreme cases, winds can exceed 100 knots. This may be double or triple the wind speed at mountaintop level. These high winds frequently lead to turbulence and wind shear at the surface, causing significant danger to aircraft and damage at the surface. Downslope windstorms often abruptly end at the "jump region," although more moderate turbulence can exist downstream. The jump region is an extremely turbulent area that can extend up to 3 $\mathrm{km}$.

**Rotors**

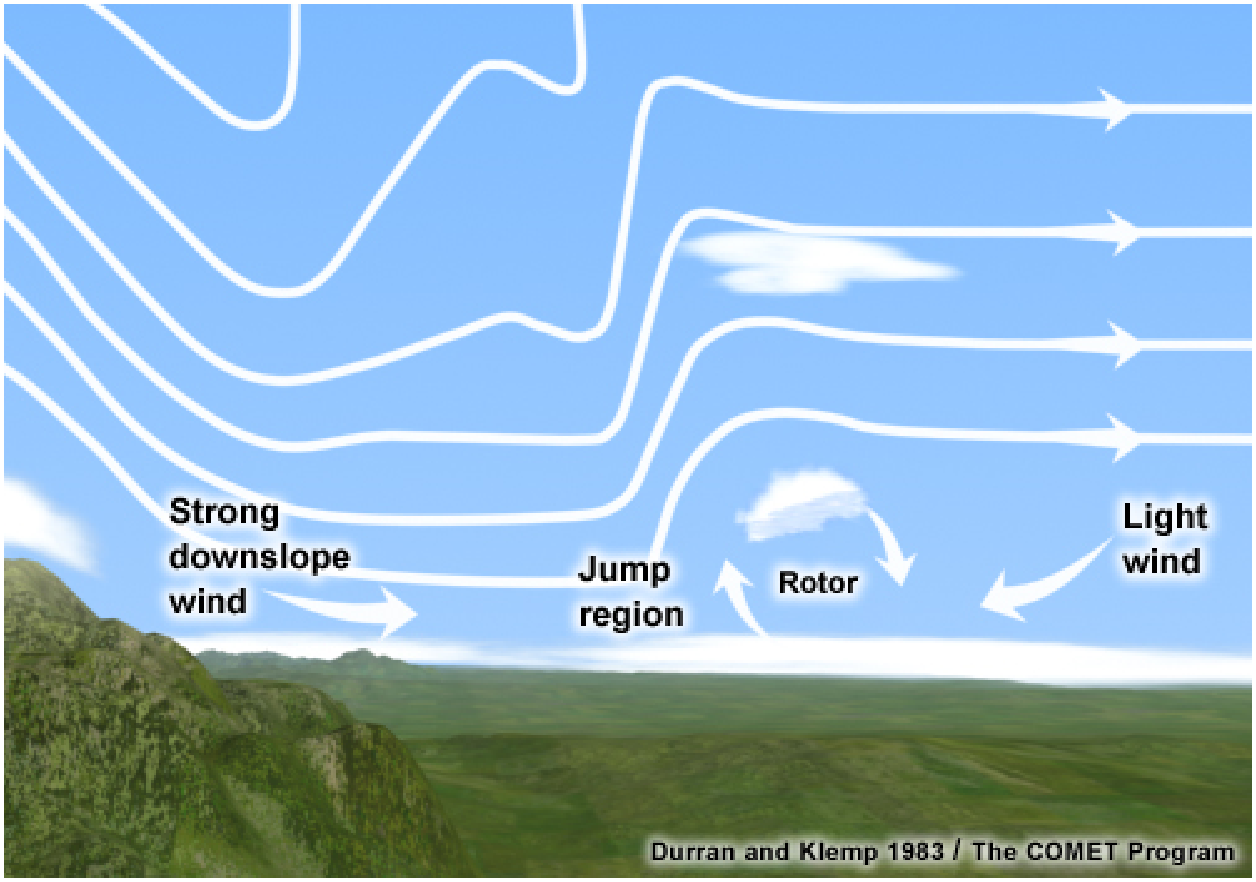


Figure 10: Typical features often associated with rotors and rotor clouds. (credit: UCAR/COMET)

Rotors are part of a low-level turbulent zone that often forms in association with a mountain wave system. Rotors are also called horizontal roll vortices because they form a complete rotational pattern, with the axis of rotation parallel to the ground. The low-level turbulent zone is another region of potentially significant turbulence. It exists immediately downstream of the jump region and under a wave crest. Rotor axes typically occur at altitude equal to or below mountain-top level and within 20 nautical miles of the ridge line. Smaller-scale rotations embedded within the low-level turbulent zone can cause rolling that exceeds an aircrafts ability to stay level. This occurs most frequently when development of a convective boundary layer aids powerful upward motions in the jump region.

Rotor location can often be identified if sufficient moisture is available to form an associated rotor cloud. Rotor clouds are found near the top of the rotor circulation and under higher lenticular clouds. Immediately above the rotor cloud, smooth, wavy air is likely.

The rotor cloud can look innocuous, but does contain strong turbulence and should be avoided by pilots. Eventually, we can expect operational NWP models to resolve rotors so that they can be identified in the absence of rotor clouds.

**Trapped Lee Waves and Clouds**

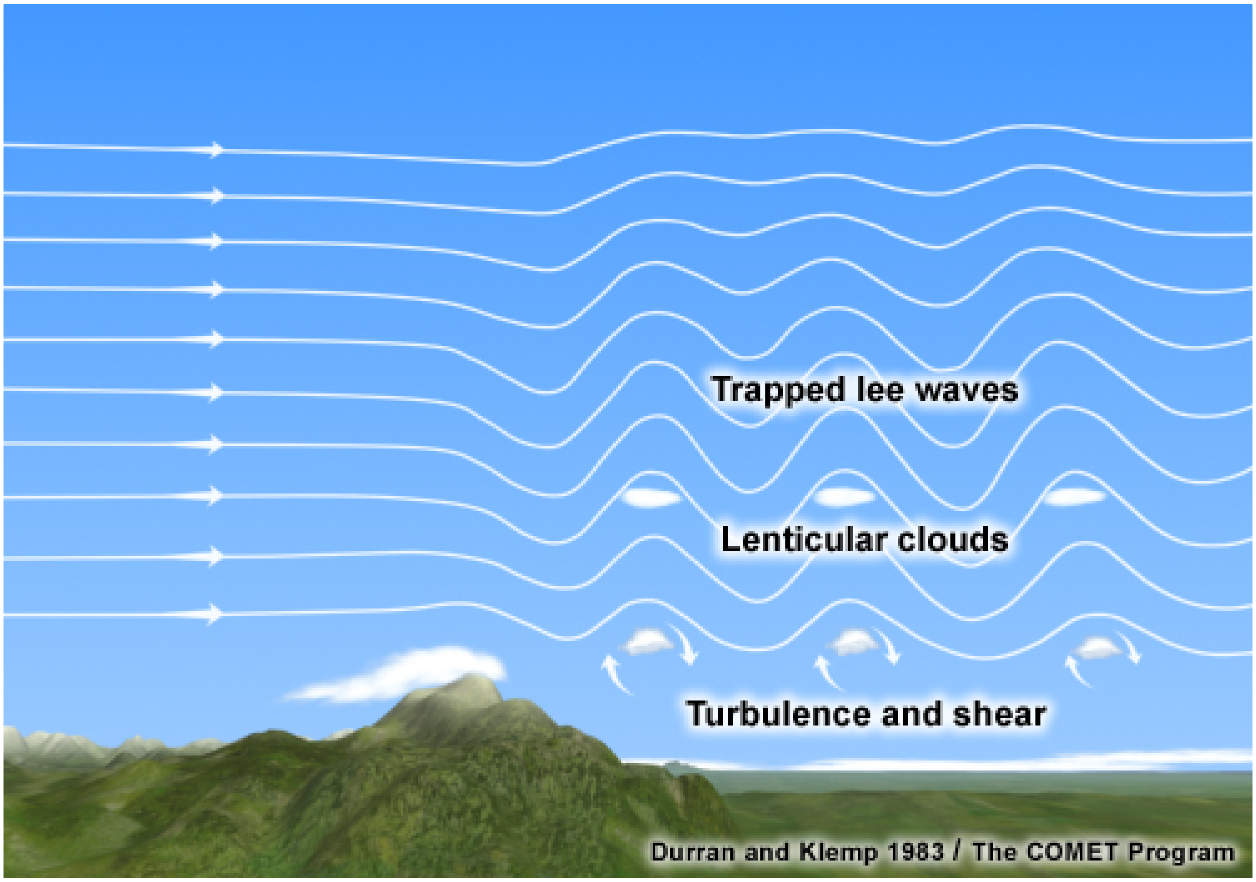


Figure 11: Typical features often associated with trapped lee waves. (credit: UCAR/COMET)

Lee waves whose energy does not propagate vertically because of strong wind shear or low stability above are said to be "trapped." Trapped lee waves are often found downstream of the rotor zone, although a weak rotor may exist under each lee wave. These waves are typically at an altitude within a few thousand feet of the mountain ridge crest and turbulence is generally restricted to altitudes below $8\ \mathrm{km}$. Strong turbulence can develop between the bases of associated lenticular clouds and the ground.

Lenticular clouds form near the crests of mountain waves. As air ascends and cools, moisture condenses, forming the cloud. As that air descends in the lee of the wave crest, the cloud evaporates. Because air flows through the cloud while the cloud itself is relatively stationary, many people refer to these clouds as standing lenticulars.

**Areal Extent of Mountain Waves**

Mountain wave activity can occur over broad regions. Figure 1 shows wave clouds covering most of Turkey, a region spanning about 1000 km! However, despite their occasionally broad extent, regions of strong or severe turbulence within mountain wave systems are often limited horizontally and vertically.

### 2.1.2 General Internal Gravity Wave Dynamics

We generally associate waves with those that appear on a water surface, *e.g.* a lake or ocean. Waves such as these are referred to as surface waves or external waves because they form along the interface between two fluids of very different densities and having their maximum amplitude at this interface (the external surface of the lower, generally homogeneous and incompressible, fluid).

In contrast, internal waves are found in a fluid with continually varying density and have their maximum amplitude within the fluid. Unfortunately, the two types of waves behave very differently, such that our intuitive understanding of external waves does little to aid our understanding of waves in the atmosphere.

Thus, we must use the equations of motion to develop our understanding of these waves. The momentum, continuity, and thermodynamic energy equations are expressed as

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_{rx} \tag{1}$$
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_{ry} \tag{2}$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz} \tag{3}$$
$$\frac{D\rho}{Dt} = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{4}$$
$$\frac{D\theta}{Dt} = \frac{\theta}{c_p T}\dot{q}\,, \tag{5}$$

where,

- $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z \rightarrow$ total derivative
- $F_{rx}$, $F_{ry}$, $F_{rz} \rightarrow$ viscous terms
- $c_p \rightarrow$ heat capacity of dry air at constant pressure
- $\dot{q} \rightarrow$ diabatic heating

To simplify the analysis, we make several assumptions:

- two-dimensional flow $\rightarrow v = 0$, $\partial/\partial y = 0$
- neglect Coriolis $\rightarrow f = 0$
- neglect viscosity $\rightarrow F_{rx}$, $F_{ry}$, $F_{rz} = 0$
- adiabatic flow $\rightarrow \dot{q} = 0$
- neglect density variations, but retain mean state variation with height $\rightarrow \rho = \overline{\rho}(z)$
- approximate buoyancy $\rightarrow g\theta'/\overline{\theta}$, where $\overline{\theta} = \overline{\theta}(z)$

The equations of motion become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial x} \tag{6}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z} + g\frac{\theta^{'}}{\overline{\theta}} \tag{7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{8}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = 0\,, \tag{9}$$

where $p^{'}$ is the departure of pressure from the hydrostatic base state ($\partial\overline{p}/\partial z = -\overline{\rho}g$) and $\theta = \overline{\theta} + \theta^{'}$.

The majority of a wave's behavior is retained even when perturbations are assumed small compared with the mean flow. Accordingly, the easiest way to demonstrate the solutions of these waves is to linearize the above equations by assuming that the flow is comprised of a mean part (assumed to be constant or to only vary with height) and a smaller amplitude fluctuating part (varies in time and space).

$$\begin{aligned} u(t,x,z) &= \overline{u}(z) + u^{'}(t,x,z) \\ w(t,x,z) &= w^{'}(t,x,z) \\ \theta(t,x,z) &= \overline{\theta}(z) + \theta^{'}(t,x,z) \end{aligned} \tag{10}$$

These terms are substituted into Eqs. (6) - (9) and any nonlinear terms (those including multiples of perturbation quantities) are neglected. By neglecting the non-linear terms, we lose lose detailed physics of the waves (*e.g.*, non-linear steepening), but retain enough information to explain their basic behavior.

The resulting linearized equations of motion are now given by

$$\frac{\partial u^{'}}{\partial t} + \overline{u}\frac{\partial u^{'}}{\partial x} + w^{'}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial x} \tag{11}$$

$$\frac{\partial w^{'}}{\partial t} + \overline{u}\frac{\partial w^{'}}{\partial x} = -\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z} + g\frac{\theta^{'}}{\overline{\theta}} \tag{12}$$

$$\frac{\partial u^{'}}{\partial x} + \frac{\partial w^{'}}{\partial z} = 0 \tag{13}$$

$$\frac{\partial \theta^{'}}{\partial t} + \overline{u}\frac{\partial \theta^{'}}{\partial x} + w^{'}\frac{\partial \overline{\theta}}{\partial z} = 0 \tag{14}$$

These equations form the basis by which we can describe various internal waves in the atmosphere.

#### 2.1.3 Two-dimensional, steady-state, adiabatic, inviscid, non-rotating, Boussinesq fluid flow over a small-amplitude mountain.

Let us consider flow over a mountain. We assume the terrain varies only in the $x$-direction, such that waves are generated in the $x$-$z$ plane. Thus, we retain the same assumptions that led to Eqs. (11) - (14). Since the forcing for the waves is stationary, we further assume a steady state ($\partial/\partial t = 0$). Our equations reduce to

$$\overline{u}\frac{\partial u^{'}}{\partial x} + w^{'}\frac{\partial \overline{u}}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial x} = 0 \tag{15}$$

$$\overline{u}\frac{\partial w^{'}}{\partial x} + \frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial z} - g\frac{\theta^{'}}{\overline{\theta}} = 0 \tag{16}$$

$$\frac{\partial u^{'}}{\partial x} + \frac{\partial w^{'}}{\partial z} = 0 \tag{17}$$

$$\overline{u}\frac{\partial \theta^{'}}{\partial x} + w^{'}\frac{N^2\overline{\theta}}{g} = 0\,, \tag{18}$$

where $N = \sqrt{(g/\overline{\theta})\partial\overline{\theta}/\partial z}$ is the Brunt-Väisälä frequency. Next, we take $\partial(Eq.\ 15)/\partial z$ and $\partial(Eq.\ 16)/\partial x$.

$$\overline{u}\frac{\partial}{\partial x}\frac{\partial u^{'}}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial^2 p^{'}}{\partial x \partial z} + w^{'}\frac{\partial^2 \overline{u}}{\partial z^2} = 0 \tag{19}$$

$$\overline{u}\frac{\partial}{\partial x}\frac{\partial w^{'}}{\partial x} + \frac{1}{\overline{\rho}}\frac{\partial^2 p^{'}}{\partial x \partial z} - \frac{g}{\overline{\theta}}\frac{\partial \theta^{'}}{\partial x} = 0\,. \tag{20}$$

Now, we subtract Eq. (19) from Eq. (20) in order to eliminate $p^{'}$, which yields

$$\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial w^{'}}{\partial x} - \frac{\partial u^{'}}{\partial z}\right) - w^{'}\frac{\partial^2 \overline{u}}{\partial z^2} - \frac{g}{\overline{\theta}}\frac{\partial \theta^{'}}{\partial x}\,. \tag{21}$$

Next, use Eq. (18) to eliminate $\partial\theta^{'}/\partial x$ :

$$\underbrace{\overline{u}\frac{\partial}{\partial x}\left(\frac{\partial w^{'}}{\partial x} - \frac{\partial u^{'}}{\partial z}\right)}_{\mathrm{I}} + \underbrace{w^{'}\frac{N^2}{\overline{u}}}_{\mathrm{II}} - \underbrace{w^{'}\frac{\partial^2 \overline{u}}{\partial z^2}}_{\mathrm{III}} = 0\,. \tag{22}$$

This may be interpreted as a vorticity equation, where (I) is the rate of change of vorticity following a fluid particle, (II) is the rate of vorticity production by buoyancy forces, and (III) is the rate of vorticity production by the vertical redistribution of the background vorticity.

To further simplify the expression, we divide Eq. (22) by $\overline{u}$ and distribute $\partial/\partial x$, which yields

$$\frac{\partial^2 w'}{\partial x^2} - \frac{\partial}{\partial z}\frac{\partial u'}{\partial x} + w'\left(\frac{N^2}{\overline{u}^2} - \frac{1}{\overline{u}}\frac{\partial^2 \overline{u}}{\partial z^2}\right) = 0\,. \tag{23}$$

Finally, use Eq. (17) to eliminate $\partial u'/\partial x$ and arrive at *Scorer's equation* (1954):

$$\nabla^2 w' + l^2(z) w' = 0\,. \tag{24}$$

Here, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is the two-dimensional Laplacian operator and $l$ is the **Scorer parameter** (Scorer 1949), which is defined by:

$$l(z) = \sqrt{\frac{N^2}{\overline{u}^2} - \frac{1}{\overline{u}}\frac{\partial^2 \overline{u}}{\partial z^2}} \tag{25}$$

Equation (24) serves as a central tool for numerous theoretical studies of small-amplitude, two-dimensional mountain waves. In the extreme case of very small Scorer parameter, e.g., when $N = 0$ and vertical shear is zero, Eq. (24) reduces to irrotational or potential flow,

$$\nabla^2 w' = 0. \tag{26}$$

If the forcing is symmetric in the basic flow direction, such as a cylinder in an unbounded fluid or a bell-shaped mountain in a half-plane, then the flow is symmetric.

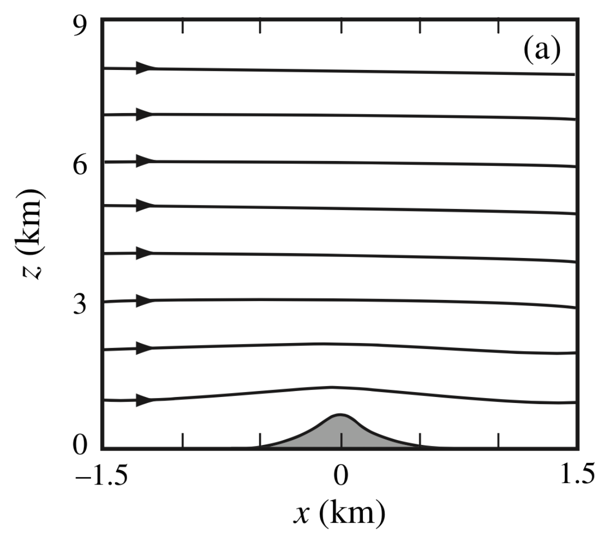


Figure 12: Streamlines of steady-state flow over an isolated, bell-shaped mountain. [From Lin (2010)]

For this particular case, there is no drag produced on the mountain since the fluid is inviscid.

### 2.1.4 Flows over two-dimensional sinusoidal mountains

We start with our equations that describe two-dimensional, steady-state, adiabatic, inviscid, non-rotating, Boussinesq fluid flow over a small-amplitude mountain. Since the terrain is sinusoidal, we assume that Eq. (24) will expect solutions in the form

$$w^{'} = \Re[\hat{w}(z)\exp(i\phi)] \,, \tag{27}$$

where $\Re[\ ]$ indicates the real part of the bracketed term (assumed hereafter), $\hat{w}(z)$ is the wave amplitude, $\phi = kx - \omega t$ is the wave phase, $k = 2\pi/L_x$ is the wavenumber of the terrain, and $L_x$ is the separation distance between ridges. Since our flow is steady state (gravity waves are stationary), then $\omega = 0$. Thus, our solution takes the form

$$w^{'} = \hat{w}(z)\exp(ikx) \,. \tag{28}$$

Note that the complex amplitude is necessary in order to fit a free-slip lower-boundary condition given any arbitrary terrain profile. For an inviscid fluid flow, this lower-boundary condition requires the flow to be tangential to the terrain (*i.e.*, the fluid particles must follow the terrain, so that the streamline slope equals the terrain slope locally). In this case, the terrain behaves according to

$$h(x) = h_m \sin kx \,, \tag{29}$$

where $h_m$ is mountain height. Then, because the flow at the lower boundary must be parallel to the boundary, the vertical velocity perturbation at the boundary is given by the rate at which the boundary height changes following the motion:

$$w^{'}(x,0) = \frac{Dh}{Dt}_{z=0} \approx \overline{u}\frac{\partial h}{\partial x} = \overline{u}kh_m \cos kx \tag{30}$$

.

Equation (30) represents the linearized lower boundary condition. At the upper boundary, we require that the flux of energy due to the perturbed flow either goes to zero as $z \to \inf$ or is directed upward from the surface (*i.e.*, no source at infinity is allowed to propagate energy into the domain).

If we substitute Eq. (28) into Eq. (24), we arrive at a modified Taylor-Goldstein equation

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \left(l^2 - k^2\right)\hat{w} = 0 \,. \tag{31}$$

We will obtain markedly different solutions depending on the relationship between $l$ and $k$.

Let's assume that the mean flow $\overline{u}(z)$ and stability $N(z)$ are both constant with height. In this case, $l^2$ reduces to $N^2/\overline{u}^2$ and we arrive at the following two-dimensional homogeneous ordinary differential equation for $\hat{w}$

$$\frac{\partial^2 \hat{w}}{\partial z^2} + m^2 \hat{w} = 0 \,, \tag{32}$$

where $m = \sqrt{(N^2/\overline{u}^2 - k^2)}$.

To solve a second-order homogeneous ODE for $y(x)$ of the generic form

$$a\frac{\partial^2 y}{\partial x^2} + b\frac{\partial y}{\partial x} + cy = 0 \,, \tag{33}$$

we look at the characteristic equation. The characteristic equation is obtained by replacing $\partial^2 y/\partial x^2$, $\partial y/\partial x$, and y with $r^2$, $r$, and $1$. This yields the easily solvable quadratic equation

$$ar^2 + br + c = 0 \,, \tag{34}$$

whose roots are given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \,. \tag{35}$$

If the roots are complex, then they can be written as $p \pm iq$, where $i$ denotes the imaginary number ($i^2 = -1$). The general form of the equation is then given by

$$y(x) = \exp(px)\left[A\exp(iqx) + B\exp(-iqx)\right] \tag{36}$$

where, $A$ and $B$ are constant complex amplitudes. If we apply this technique to Eq. (32), then $a = 1$, $b = 0$, and $c = m^2$, meaning the roots are $\pm im$. The the solution is given by

$$\hat{w}(z) = A\exp(imz) + B\exp(-imz) \,. \tag{37}$$

Substitution of Eq. (37) in Eq. (28) yields the solution for $w^{'}$, given by

$$w^{'} = A\exp(i[kx + mz]) + B\exp(i[kx - mz]) \,. \tag{38}$$

There are two possible cases from Eq.(38): when $m$ is real and when $m$ is imaginary. We will discuss these in the next class period, plus examine flows with other forcing mechanisms.