LES of Turbulent Flows: Lectures 8 and 9

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Overview

1. LES filtered equations for compressible flows

LES filtered equations for compressible flows

- What if we apply a filter to the compressible Navier-Stokes equations? Let's look at mass continuity as an example.

$$\widetilde{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i}} = 0$$

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$$\frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial \widetilde{\rho u_i}}{\partial x_i} = 0$$

- We end up with a SFS term involving density, $\widetilde{\rho u_i}$, which requires modeling.
- How do we avoid this?

Density-weighted filtering

- Density-weighted filtering was formalized by Favre (1983) for ensemble statistics.
- The procedure is often referred to as Favre filtering.
- For an arbitrary quantity ϕ, the Favre filter is applied as:

$$\overline{\phi} = \frac{\widetilde{\rho\phi}}{\widetilde{\rho}} \Rightarrow \widetilde{\rho\phi} = \widetilde{\rho}\overline{\phi}$$

- As we approach incompressibility, $\overline{\phi} \sim \widetilde{\phi}$
- For LES purposes, the Favre filter represents a density-weighted spatial average.

Non-dimensional compressible equations of motion

- Non-dimensionalize

$$
\begin{aligned}
u_i^* &= \frac{u_i}{U} \\
x_i^* &= \frac{x_i}{\ell} \\
p^* &= \frac{p}{\rho_o U^2} = \frac{p\ell}{\mu_o U} \\
t^* &= \frac{tU}{\ell} \\
\theta^* &= \frac{\theta}{\theta_o} \\
\nu^* &= \frac{\nu}{\nu_o} \\
\mu^* &= \frac{\mu}{\mu_o}
\end{aligned}
$$

,

where the * denotes a non-dimensionalized term.

Favre-filtered conservation of mass

- Apply Favre filter to mass continuity:

$$\widetilde{\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i}} = 0 \Rightarrow \frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial \widetilde{\rho u_i}}{\partial x_i} = 0$$

- Recall that the Favre filter is applied as: $\widetilde{\rho\phi} = \widetilde{\rho}\overline{\phi}$.
- Substitution yields the Favre-filtered mass continuity equation:

$$\frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial (\widetilde{\rho}\,\overline{u_i})}{\partial x_i} = 0$$

Favre-filtered conservation of mass

- Next, let's rewrite in non-dimensional terms

$$\frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial (\widetilde{\rho}\overline{\phi})}{\partial x_i} = \cancel{\frac{\rho_o U}{\ell}} \frac{\partial \widetilde{\rho}^*}{\partial t^*} + \cancel{\frac{\rho_o U}{\ell}} \frac{\partial \widetilde{\rho}^* \overline{u_i^*}}{\partial x_i^*} = 0$$

The result is the non-dimensional Favre-filtered mass continuity equation

$$\boxed{\frac{\partial \widetilde{\rho}}{\partial t} + \frac{\partial (\widetilde{\rho}\,\overline{u_i})}{\partial x_i} = 0}$$

where we have dropped the * notation for convenience.

Favre-filtered conservation of momentum

- Start with the conservation of mass:

$$\underbrace{\frac{\partial(\rho u_i)}{\partial t}}_{1} + \underbrace{\frac{\partial(\rho u_i u_j)}{\partial x_j}}_{2} = -\underbrace{\frac{\partial p}{\partial x_i}}_{3} + \underbrace{\frac{\partial}{\partial x_j}\left[2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]}_{4}$$

- To make life easier, let's filter each term individually.

Favre-filtered conservation of momentum

- Term 1

$$\widetilde{\frac{\partial(\rho u_i)}{\partial t}} = \frac{\partial(\widetilde{\rho u_i})}{\partial t} = \frac{\partial(\widetilde{\overline{\rho u_i}})}{\partial t}$$

Non-dimensionalizing yields

$$\frac{\partial(\widetilde{\overline{\rho u_i}})}{\partial t} = \frac{(\partial \widetilde{\rho^*} \rho_o \overline{u_i^*} U)}{\partial t^* \ell / U} = \boxed{\frac{\rho_o U^2}{\ell} \frac{\partial(\widetilde{\overline{\rho u_i}})}{\partial t}}$$

where we have dropped the * notation for convenience.

Favre-filtered conservation of momentum

- Term 2

$$\widetilde{\frac{\partial(\rho u_i u_j)}{\partial x_j}} = \frac{\partial(\widetilde{\rho u_i u_j})}{\partial x_j} = \frac{\partial(\widetilde{\rho}\,\overline{u_i u_j})}{\partial x_j}$$

Recall from our derivation of the incompressible momentum equation that we utilized the Leonard (1974) decomposition:

$$\widetilde{u_i u_j} = \widetilde{u}_i \widetilde{u}_j + \tau_{ij}$$

where τ_{ij} is the SFS stress tensor. Similarly

$$\overline{u_i u_j} = \overline{u_i}\,\overline{u_j} + \tau_{ij}$$

where we have dropped the * notation for convenience.

Favre-filtered conservation of momentum

- Term 2

  Substitution yields

$$\frac{\partial(\widetilde{\rho}\,\overline{u_i u_j})}{\partial x_j} = \frac{\partial\widetilde{\rho}(\overline{u_i}\,\overline{u_j} + \tau_{ij})}{\partial x_j} = \frac{\partial(\widetilde{\rho}\,\overline{u_i}\,\overline{u_j})}{\partial x_j} + \frac{\partial(\widetilde{\rho}\,\tau_{ij})}{\partial x_j}$$

  Non-dimensionalizing yields

$$\frac{\partial(\widetilde{\rho}\,\overline{u_i}\,\overline{u_j})}{\partial x_j} + \frac{\partial(\widetilde{\rho}\,\tau_{ij})}{\partial x_j} = \frac{\partial(\widetilde{\rho^*}\rho_o\,\overline{u_i^*}U\,\overline{u_j^*}U)}{\partial x_j^*\ell} + \frac{\partial(\widetilde{\rho^*}\rho_o\,\tau_{ij}^* U^2)}{\partial x_j \ell}$$

$$= \boxed{\frac{\rho_o U^2}{\ell}\left[\frac{\partial(\widetilde{\rho}\,\overline{u_i}\,\overline{u_j})}{\partial x_j} + \frac{\partial(\widetilde{\rho}\,\tau_{ij})}{\partial x_j}\right]}$$

  where we have dropped the * notation for convenience.

Favre-filtered conservation of momentum

- Term 3

$$\widetilde{\frac{\partial p}{\partial x_i}} = \frac{\partial \widetilde{p}}{\partial x_i}$$

Non-dimensionalizing yields

$$\frac{\partial \widetilde{p}}{\partial x_i} = \frac{\partial \widetilde{p^*} \rho_o U^2}{\partial x_i^* \ell} = \boxed{\frac{\rho_o U^2}{\ell} \frac{\partial \widetilde{p}}{\partial x_i}}$$

where we have dropped the * notation for convenience.

Favre-filtered conservation of momentum

- Term 4

$$\frac{\partial}{\partial x_j}\left[2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]$$

First note that μ is not constant, but rather a function of temperature. So, $\mu = \mu(T)$ and $\bar{\mu} = \mu(\bar{T})$. If we define

$$\sigma_{ij} = 2\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right),$$

then we can rewrite Term 4 as

$$\frac{\partial \sigma_{ij}}{\partial x_j}$$

Favre-filtered conservation of momentum

- Term 4

$$\widetilde{\frac{\partial \sigma_{ij}}{\partial x_j}} = \frac{\partial \widetilde{\sigma_{ij}}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\widetilde{2\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)}\right]$$

Thus, $\widetilde{\sigma_{ij}}$ cannot be directly expressed in terms of the basic filtered variables

We decompose to $\widetilde{\sigma_{ij}}$ into a smooth flow contribution and a subfilter contribution.

Favre-filtered conservation of momentum

- Term 4

$$\text{smooth} \Rightarrow \overline{\sigma_{ij}} = 2\bar{\mu}\left(\bar{S}_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial \overline{u_k}}{\partial x_k}\right)$$

$$\text{subgrid} \Rightarrow \widetilde{\sigma_{ij}} - \overline{\sigma_{ij}}$$

Thus,

$$\frac{\partial \widetilde{\sigma_{ij}}}{\partial x_j} = \frac{\partial \overline{\sigma_{ij}}}{\partial x_j} + \frac{\partial\left(\widetilde{\sigma_{ij}} - \overline{\sigma_{ij}}\right)}{\partial x_j}$$

Let's non-dimensionalize

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[2\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\right]$$

and apply to all of the terms in the above expression.

Favre-filtered conservation of momentum

- Term 4
  Recall that Re$=U\ell/\nu = U\ell\rho_o/\mu_o \Rightarrow \mu_o = U\ell\rho_o/$Re

$$\mu = \mu^*\mu_o = \frac{\mu^* U\ell\rho_o}{\text{Re}}$$

and

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i^* U}{\partial x_j^* \ell} + \frac{\partial u_j^* U}{\partial x_i^* \ell}\right) = \frac{U}{\ell}\frac{1}{2}\left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*}\right) = \frac{U}{\ell}S_{ij}^*$$

and

$$\frac{\partial u_k}{\partial x_k} = \frac{\partial u_k^* U}{\partial x_k^* \ell} = \frac{U}{\ell}\frac{\partial u_k^*}{\partial x_k^*}$$

Favre-filtered conservation of momentum

- Term 4

  Putting them altogether yields

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{1}{\ell}\frac{\partial}{\partial x_j^*}\left[2\frac{\mu^* U \ell \rho_o}{\text{Re}}\left(\frac{U}{\ell}\left\{S_{ij}^* - \frac{1}{3}\delta_{ij}\frac{\partial u_k^*}{\partial x_k^*}\right\}\right)\right]$$
$$= \frac{\rho_o U^2}{\ell}\frac{\partial}{\partial x_j^*}\left[\frac{2\mu^*}{\text{Re}}\left(S_{ij}^* - \frac{1}{3}\delta_{ij}\frac{\partial u_k^*}{\partial x_k^*}\right)\right]$$
$$= \frac{\rho_o U^2}{\ell}\frac{\partial \sigma_{ij}^*}{\partial x_j^*}$$

Thus,

$$\boxed{\frac{\partial \widetilde{\sigma_{ij}}}{\partial x_j} = \frac{\rho_o U^2}{\ell}\frac{\partial \overline{\sigma_{ij}}}{\partial x_j} + \frac{\rho_o U^2}{\ell}\frac{\partial\left(\widetilde{\sigma_{ij}} - \overline{\sigma_{ij}}\right)}{\partial x_j}}$$

where we have dropped the * notation for convenience.

Favre-filtered conservation of momentum

- Finally, we combine Terms 1-4 and group SFS terms

$$\frac{\rho_o U^2}{\ell}\frac{\partial(\widetilde{\rho}\overline{u_i})}{\partial t} + \frac{\rho_o U^2}{\ell}\frac{\partial(\widetilde{\rho}\,\overline{u_i}\,\overline{u_j})}{\partial x_j} + \frac{\rho_o U^2}{\ell}\frac{\partial \widetilde{p}}{\partial x_i} - \frac{\rho_o U^2}{\ell}\frac{\partial \overline{\sigma_{ij}}}{\partial x_j}$$

$$= -\frac{\rho_o U^2}{\ell}\frac{\partial(\widetilde{\rho}\,\tau_{ij})}{\partial x_j} + \frac{\rho_o U^2}{\ell}\frac{\partial\left(\widetilde{\sigma_{ij}} - \overline{\sigma_{ij}}\right)}{\partial x_j}$$

Cancelling $\rho_o U^2/\ell$ yields the dimensionless Favre-filtered conservation of momentum equation

$$\boxed{\frac{\partial(\widetilde{\rho}\overline{u_i})}{\partial t} + \frac{\partial(\widetilde{\rho}\,\overline{u_i}\,\overline{u_j})}{\partial x_j} + \frac{\partial \widetilde{p}}{\partial x_i} - \frac{\partial \overline{\sigma_{ij}}}{\partial x_j} = -\frac{\partial(\widetilde{\rho}\,\tau_{ij})}{\partial x_j} + \frac{\partial\left(\widetilde{\sigma_{ij}} - \overline{\sigma_{ij}}\right)}{\partial x_j}}$$

Favre-filtered conservation of momentum

$$\frac{\partial(\widetilde{\rho}\overline{u_i})}{\partial t} + \frac{\partial(\widetilde{\rho}\,\overline{u_i}\,\overline{u_j})}{\partial x_j} + \frac{\partial \widetilde{p}}{\partial x_i} - \frac{\partial \overline{\sigma_{ij}}}{\partial x_j} = -\frac{\partial(\widetilde{\rho}\,\tau_{ij})}{\partial x_j} + \frac{\partial\left(\widetilde{\sigma_{ij}} - \overline{\sigma_{ij}}\right)}{\partial x_j}$$

The terms comprising the SFS viscous term on the RHS are:

- "smooth" viscous stress tensor

$$\overline{\sigma_{ij}} = \frac{2}{\mathsf{Re}}\bar{\mu}\left(\bar{S}_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial \overline{u_k}}{\partial x_k}\right)$$

- non-linear viscous stress tensor

$$\widetilde{\sigma_{ij}} = \frac{2}{\mathsf{Re}}\widetilde{\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)}$$

Favre-filtered kinetic energy equation

- The full kinetic energy equation is written as

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_j}\left[(e+p)u_j\right] - \frac{\partial (u_i \sigma_{ij})}{\partial x_j} + \frac{\partial q_j}{\partial x_j} = 0$$

- e is the total energy density

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u_i u_i \qquad \text{where} \qquad \gamma = \frac{c_p}{c_v} \approx 1.4 \text{ for air}$$

- σ_{ij} is the viscous stress tensor

$$\frac{2\mu}{\mathsf{Re}}\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

Favre-filtered kinetic energy equation

- The full kinetic energy equation is written as

$$\frac{\partial e}{\partial t} + \frac{\partial([e+p)u_j])}{\partial x_j} - \frac{\partial(u_i \sigma_{ij})}{\partial x_j} + \frac{\partial q_j}{\partial x_j} = 0$$

- q_j is the heat flux

$$q_j = -\frac{\mu}{(\gamma - 1)\mathsf{RePrM}^2}\frac{\partial T}{\partial x_j}$$

where Pr is the Prandtl number (≈ 0.72 for air) and M is the reference Mach number:

$$M = \frac{U}{c} \qquad \text{where} \qquad c = \sqrt{\gamma R T}$$

and R is the ideal gas law constant.

Favre-filtered kinetic energy equation

- Applying the Favre filter to this equation is left as an exercise for you.
- The resulting Favre-filtered kinetic energy equation is given by

$$\boxed{\begin{aligned}\frac{\partial \bar{e}}{\partial t} + \frac{\partial\left[(\bar{e}+\tilde{p})\overline{u_j}\right]}{\partial x_j} - \frac{(\overline{u_i}\,\overline{\sigma_{ij}})}{\partial x_j} + \frac{\partial \overline{q_j}}{\partial x_j} = &- a_1 - a_2 \\ &- a_3 + a_4 + a_5 - a_6\end{aligned}}$$

Favre-filtered kinetic energy equation

$$\frac{\partial \bar{e}}{\partial t} + \frac{\partial \left[(\bar{e} + \tilde{p})\overline{u_j}\right]}{\partial x_j} - \frac{(\overline{u_i}\,\overline{\sigma_{ij}})}{\partial x_j} + \frac{\partial \overline{q_j}}{\partial x_j} = - a_1 - a_2 - a_3 + a_4 + a_5 - a_6$$

$$a_1 = \overline{u_i}\frac{\partial(\widetilde{p}\tau_{ij})}{\partial x_j}$$ ⇒ kinetic energy transferred from resolved to SFSs

$$a_2 = \frac{1}{\gamma - 1}\frac{\widetilde{pu_j} - \widetilde{p}\overline{u_j}}{\partial x_j}$$ ⇒ pressure velocity SFS term (effect of SFS turbulence on the conduction of heat at resolved scales)

$$a_3 = \widetilde{p\frac{\partial u_j}{\partial x_j}} - \widetilde{p}\frac{\partial \overline{u_j}}{\partial x_j}$$ ⇒ compressibility effects (vanishes for incompressible)

Favre-filtered kinetic energy equation

$$\frac{\partial \bar{e}}{\partial t} + \frac{\partial\left[(\bar{e}+\tilde{p})\overline{u_j}\right]}{\partial x_j} - \frac{(\overline{u_i}\,\overline{\sigma_{ij}})}{\partial x_j} + \frac{\partial \overline{q_j}}{\partial x_j} = -a_1 - a_2 - a_3 + a_4 + a_5 - a_6$$

$$a_4 = \widetilde{\sigma_{ij}\frac{\partial u_i}{\partial x_j}} - \widetilde{\sigma_{ij}}\frac{\partial \overline{u_i}}{\partial x_j}$$ ⇒ conversion of SFS kinetic energy to internal energy by viscous dissipation

$$a_5 = \frac{\partial(\widetilde{\sigma_{ij}}\overline{u_i} - \overline{\sigma_{ij}}\,\overline{u_i})}{\partial x_j}$$ ⇒ SFS viscous stress term

$$a_6 = \frac{\partial(\widetilde{q_j} - \overline{q_j})}{\partial x_j}$$ ⇒ SFS heat flux term

It is typically assumed that $\widetilde{\sigma_{ij}} - \overline{\sigma_{ij}} \approx 0$ and $\widetilde{q_j} - \overline{q_j} \approx 0$. This eliminates terms a_5 and a_6.