LES of Turbulent Flows: Lecture 7

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Overview

1. Recap of LES filtered equations for incompressible flows
2. The filtered kinetic energy equation for incompressible equations

LES filtered equations for incompressible flows

Mass

- $\frac{\partial \tilde{u}_i}{\partial x_i} = 0$

Momentum

- $\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial (\tilde{u}_i \tilde{u}_j)}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j} + F_i$

Scalar

- $\frac{\partial \tilde{\theta}}{\partial t} + \frac{\partial (\tilde{u}_i \tilde{\theta})}{\partial x_j} = \frac{1}{Sc\ Re} \frac{\partial^2 \tilde{\theta}}{\partial x_j^2} - \frac{\partial q_i^r}{\partial x_j} + Q$

SFS stress

- $\tau_{ij}^r = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j$

SFS flux

- $q_i^r = \widetilde{u_i \theta} - \tilde{u}_i \tilde{\theta}$

Up next, turbulence kinetic energy

- We've talked about variance (or energy) when discussing turbulence and filtering.
- When we examined the application of the LES filter at scale Δ we saw the effect of the filter on the distribution of energy with scale.
- A natural way to extend our examination of scale separation and energy is to look at the evolution of the filtered variance or turbulence kinetic energy.

The filtered kinetic energy equation

- We can define the total filtered kinetic energy as

$$\tilde{E} = \frac{1}{2}\widetilde{u_i u_i}$$

- Next we decompose this in the standard way as:

$$\tilde{E} = \underbrace{\tilde{E}_f}_{\substack{\text{Resolved}\\\text{kinetic}\\\text{energy}}} + \underbrace{k_r}_{\substack{\text{SFS}\\\text{kinetic}\\\text{energy}}}$$

- The SFS kinetic energy (residual kinetic energy) is defined as:

$$k_r = \frac{1}{2}\left(\widetilde{u_i u_i} - \tilde{u}_i \tilde{u}_i\right)$$

The filtered kinetic energy equation

- Putting them together yields:

$$\frac{1}{2}\widetilde{u_i u_j} = \tilde{E}_f + \frac{1}{2}\left(\widetilde{u_i u_i} - \tilde{u}_i \tilde{u}_i\right)$$

Thus the resolved (filtered) kinetic energy is then given by:

$$\tilde{E}_f = \frac{1}{2}\tilde{u}_i \tilde{u}_i$$

The filtered kinetic energy equation

- We can develop an equation for $\tilde{E}_f$ by multiplying the filtered LES momentum equation by $\tilde{u}_i$:

$$\underbrace{\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t}}_{1} + \underbrace{\tilde{u}_i \frac{\partial (\tilde{u}_i \tilde{u}_j)}{\partial x_j}}_{2} = \underbrace{-\tilde{u}_i \frac{\partial \tilde{p}}{\partial x_i}}_{3} + \underbrace{\frac{\tilde{u}_i}{Re} \frac{\partial^2 \tilde{u}_i}{\partial x_j^2}}_{4} \underbrace{-\tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j}}_{5}$$

- The gist: we want to write the equation in terms of $\tilde{E}_f$. First we focus on terms (1)-(3) by applying the product rule, *i.e.*

$$\frac{\partial (ab)}{\partial x} = a \frac{\partial b}{\partial x} + b \frac{\partial a}{\partial x}$$

- Recall that the filtered rate of strain tensor is given by

$$\tilde{S}_{ij} = \frac{1}{2} \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right)$$

The filtered kinetic energy equation

Term 1:

$$\frac{\partial(\tilde{u}_i\tilde{u}_i)}{\partial t} = \tilde{u}_i\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_i\frac{\partial \tilde{u}_i}{\partial t} = 2\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial t}$$

$$\Rightarrow \tilde{u}_i\frac{\partial \tilde{u}_i}{\partial t} = \frac{1}{2}\frac{\partial(\tilde{u}_i\tilde{u}_i)}{\partial t}$$

$$\boxed{\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial t} = \frac{\partial \tilde{E}_f}{\partial t}}$$

The filtered kinetic energy equation

Term 2:

$$\text{first:} \qquad \tilde{u}_i \frac{\partial(\tilde{u}_i \tilde{u}_j)}{\partial x_j} = \tilde{u}_i \tilde{u}_i \cancelto{0 \text{ via continuity}}{\frac{\partial \tilde{u}_j}{\partial x_j}} + \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}$$

$$\text{next:} \qquad \tilde{u}_j \frac{\partial(\tilde{u}_i \tilde{u}_i)}{\partial x_j} = \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = 2 \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}$$

$$\Rightarrow u_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \frac{1}{2} \tilde{u}_j \frac{\partial(\tilde{u}_i \tilde{u}_i)}{\partial x_j}$$

$$\boxed{\tilde{u}_i \frac{\partial(\tilde{u}_i \tilde{u}_j)}{\partial x_j} = \tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j}}$$

The filtered kinetic energy equation

Term 3:

$$\frac{\partial(\tilde{u}_i\tilde{p})}{\partial x_i} = \tilde{p}\underbrace{\cancel{\frac{\tilde{u}_i}{\partial x_i}}}_{0 \text{ via continuity}} + \tilde{u}_i\frac{\partial \tilde{p}}{\partial x_i}$$

$$\Rightarrow \boxed{\tilde{u}_i\frac{\partial \tilde{p}}{\partial x_j} = \frac{\partial(\tilde{u}_i\tilde{p})}{\partial x_i}}$$

The filtered kinetic energy equation

Term 4:

$$\frac{\partial^2(\tilde{u}_i\tilde{u}_i)}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left[\frac{\partial(\tilde{u}_i\tilde{u}_i)}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\left[\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial x_j} + u_i\frac{\partial \tilde{u}_i}{\partial x_j}\right]$$

$$\Rightarrow \frac{\partial}{\partial x_j}\left[2\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial x_j}\right] = 2\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j} + 2\tilde{u}_i\frac{\partial^2 \tilde{u}_i}{\partial x_j^2}$$

$$\frac{\tilde{u}_i}{\mathrm{Re}}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} = \underbrace{\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left[\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial x_j}\right]}_{\text{4a}} - \underbrace{\frac{1}{\mathrm{Re}}\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j}}_{\text{4b}}$$

The filtered kinetic energy equation

Term 4a:

$$\frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left[\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial x_j}\right] = \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left[\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_i\frac{\partial \tilde{u}_j}{\partial x_i} - \tilde{u}_i\frac{\partial \tilde{u}_j}{\partial x_i}\right]$$

$$= \frac{1}{\mathrm{Re}}\frac{\partial(2\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left[\tilde{u}_i\frac{\partial \tilde{u}_j}{\partial x_i}\right]$$

$$= \frac{1}{\mathrm{Re}}\frac{\partial(2\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - \frac{1}{\mathrm{Re}}\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_j}{\partial x_i} - \frac{\tilde{u}_i}{\mathrm{Re}}\cancelto{0 \text{ via continuity}}{\frac{\partial^2 \tilde{u}_j}{\partial x_j \partial x_i}}$$

$$= \frac{1}{\mathrm{Re}}\frac{\partial(2\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - \frac{1}{\mathrm{Re}}\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_j}{\partial x_i}$$

The filtered kinetic energy equation

Combine terms 4a and 4b:

$$\frac{\tilde{u}_i}{\mathsf{Re}}\frac{\partial^2\tilde{u}_i}{\partial x_j^2} = \frac{1}{\mathsf{Re}}\left[\frac{\partial(2\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - \frac{\partial\tilde{u}_i}{\partial x_j}\frac{\partial\tilde{u}_j}{\partial x_i} - \frac{\partial\tilde{u}_i}{\partial x_j}\frac{\partial\tilde{u}_i}{\partial x_j}\right]$$

This looks crazy. How do we reduce it further?

The filtered kinetic energy equation

Tensors:

- Recall that any arbitrary 2nd-order tensor can be written as the sum of a symmetric and an antisymmetric tensor:

$$\underbrace{B_{ij}}_{\text{arbitrary}} = \underbrace{S_{ij}}_{\text{symmetric}} + \underbrace{A_{ij}}_{\text{antisymmetric}}$$

- Symmetric

$$S_{ij} = S_{ji}$$

- Antisymmetric

$$A_{ij} = -A_{ji}$$

The filtered kinetic energy equation

Tensors:

- Proof:

$$B_{ij} = \frac{1}{2}B_{ij} + \frac{1}{2}B_{ij} = \underbrace{\frac{1}{2}\left(B_{ij} + B_{ji}\right)}_{S_{ij}} + \underbrace{\frac{1}{2}\left(B_{ij} - B_{ji}\right)}_{A_{ij}}$$

$$S_{ij} = \frac{1}{2}\left(B_{ij} + B_{ji}\right) = \frac{1}{2}\left(B_{ji} + B_{ij}\right) = S_{ji}$$

$$A_{ij} = \frac{1}{2}\left(B_{ij} - B_{ji}\right) = -\frac{1}{2}\left(B_{ji} - B_{ij}\right) = -A_{ji}$$

The filtered kinetic energy equation

Tensors:

- Let's apply this tensor idea

$$\frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{S}_{ij} + \tilde{A}_{ij} \qquad \text{and} \qquad \frac{\partial \tilde{u}_j}{\partial x_i} = \tilde{S}_{ji} + \tilde{A}_{ji}$$

- Using this

$$\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{S}_{ij}\tilde{S}_{ij} + 2\tilde{S}_{ij}\tilde{A}_{ij} + \tilde{A}_{ij}\tilde{A}_{ij}$$

$$\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_j}{\partial x_i} = \tilde{S}_{ij}\tilde{S}_{ji} + \tilde{S}_{ij}\tilde{A}_{ji} + \tilde{S}_{ji}\tilde{A}_{ij} + \tilde{A}_{ij}\tilde{A}_{ji}$$

- Whoa, this is getting out of hand. Can we reduce these expressions?

The filtered kinetic energy equation

Tensors:

- Let's look at a general property of symmetric/antisymmetric tensors:

  $$S_{ij}A_{ij} = -S_{ij}Aji$$

  Rename dummy variables and use symmetry

  $$-S_{ij}Aji = -S_{ji}Aij = -S_{ij}Aij$$

  Thus

  $$S_{ij}A_{ij} = -S_{ij}Aij \Rightarrow 2S_{ij}A_{ij} = 0 \Rightarrow S_{ij}A_{ij} = 0$$

  The product of a symmetric and antisymmetric tensor is 0.

The filtered kinetic energy equation

Tensors:

- Back to our definitions

$$\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{S}_{ij}\tilde{S}_{ij} + \cancelto{0}{2\tilde{S}_{ij}\tilde{A}_{ij}} + \tilde{A}_{ij}\tilde{A}_{ij} = \tilde{S}_{ij}\tilde{S}_{ij} + \tilde{A}_{ij}\tilde{A}_{ij}$$

$$\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_j}{\partial x_i} = \tilde{S}_{ij}\tilde{S}_{ji} + \cancelto{0}{\tilde{S}_{ij}\tilde{A}_{ji}} + \cancelto{0}{\tilde{S}_{ji}\tilde{A}_{ij}} + \tilde{A}_{ij}\tilde{A}_{ji} = \tilde{S}_{ij}\tilde{S}_{ji} + \tilde{A}_{ij}\tilde{A}_{ji}$$

Let's plug these expressions back into Term 4.

The filtered kinetic energy equation

Term 4:

$$\begin{aligned}\frac{\tilde{u}_i}{\mathrm{Re}}\frac{\partial^2\tilde{u}_i}{\partial x_j^2} &= \frac{1}{\mathrm{Re}}\left[\frac{\partial(2\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - \frac{\partial\tilde{u}_i}{\partial x_j}\frac{\partial\tilde{u}_j}{\partial x_i} - \frac{\partial\tilde{u}_i}{\partial x_j}\frac{\partial\tilde{u}_i}{\partial x_j}\right]\\ &= \frac{1}{\mathrm{Re}}\left[\frac{\partial(2\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - S_{ij}\tilde{S}_{ij} - \tilde{A}_{ij}\tilde{A}_{ij} - \tilde{S}_{ij}\tilde{S}_{ji} - \tilde{A}_{ij}\tilde{A}_{ji}\right]\end{aligned}$$

Note $S_{ji} = S_{ij}$ and $\tilde{A}_{ij}\tilde{A}_{ji} = -\tilde{A}_{ij}\tilde{A}_{ij}$

$$= \frac{1}{\mathrm{Re}}\left[\frac{\partial(2\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - S_{ij}\tilde{S}_{ij} - \cancel{\tilde{A}_{ij}\tilde{A}_{ij}} - \tilde{S}_{ij}\tilde{S}_{ij} + \cancel{\tilde{A}_{ij}\tilde{A}_{ij}}\right]$$

$$\boxed{= \frac{2}{\mathrm{Re}}\frac{\partial(\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - \underbrace{\frac{2}{\mathrm{Re}}\tilde{S}_{ij}\tilde{S}_{ij}}_{\epsilon_f}}$$

The filtered kinetic energy equation

- Term 5:

$$\frac{\partial(\tilde{u}_i \tau_{ij})}{\partial x_j} = \tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij} \frac{\partial \tilde{u}_i}{\partial x_j}$$

$$\boxed{\Rightarrow \tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial(\tilde{u}_i \tau_{ij})}{\partial x_j} \underbrace{-\tau_{ij} \frac{\partial \tilde{u}_i}{\partial x_j}}_{\Pi = \tau_{ij} \tilde{S}_{ij}}}$$

Note:

$$\tau_{ij} \frac{\partial \tilde{u}_i}{\partial x_j} = \tau_{ij}(\tilde{S}_{ij} + \tilde{A}_{ij}) = \tau_{ij}\tilde{S}_{ij} + \tau_{ij}\tilde{A}_{ij}$$

$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j$ is symmetric

$$= \tau_{ij}\tilde{S}_{ij} + \cancel{\tau_{ij}\tilde{A}_{ij}}$$

$$= \tau_{ij}\tilde{S}_{ij}$$

Now, let's put terms (1)-(5) back together.

The filtered kinetic energy equation

The dimensionless filtered TKE equation

$$\underbrace{\frac{\partial \tilde{E}_f}{\partial t}}_{\mathrm{I}} + \underbrace{\tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j}}_{\mathrm{II}} = -\underbrace{\frac{\partial(\tilde{u}_i \tilde{p})}{\partial x_i}}_{\mathrm{III}} - \underbrace{\frac{\partial(\tilde{u}_i \tau_{ij})}{\partial x_j}}_{\mathrm{IV}} - \underbrace{\frac{2}{\mathrm{Re}} \frac{\partial(\tilde{u}_i \tilde{S}_{ij})}{\partial x_j}}_{\mathrm{V}} - \underbrace{\epsilon_f}_{\mathrm{VI}} - \underbrace{\Pi}_{\mathrm{VII}}$$

- I "storage" of $\tilde{E}_f$
- II advection of $\tilde{E}_f$
- III pressure transport
- IV transport of SFS stress τ_{ij}
- V transport of viscous stress
- VI dissipation by viscous stress
- VII SFS dissipation

The filtered kinetic energy equation

The dimensional filtered TKE equation

$$\underbrace{\frac{\partial \tilde{E}_f}{\partial t}}_{\text{I}} + \underbrace{\tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j}}_{\text{II}} = -\underbrace{\frac{1}{\rho}\frac{\partial(\tilde{u}_i \tilde{p})}{\partial x_i}}_{\text{III}} - \underbrace{\frac{\partial(\tilde{u}_i \tau_{ij})}{\partial x_j}}_{\text{IV}} - \underbrace{2\nu \frac{\partial(\tilde{u}_i \tilde{S}_{ij})}{\partial x_j}}_{\text{V}} - \underbrace{\epsilon_f}_{\text{VI}} - \underbrace{\Pi}_{\text{VII}}$$

where $\epsilon_f = 2\nu \tilde{S}_{ij}\tilde{S}_{ij}$.

- I “storage” of $\tilde{E}_f$
- II advection of $\tilde{E}_f$
- III pressure transport
- IV transport of SFS stress τ_{ij}
- V transport of viscous stress
- VI dissipation by viscous stress
- VII SFS dissipation

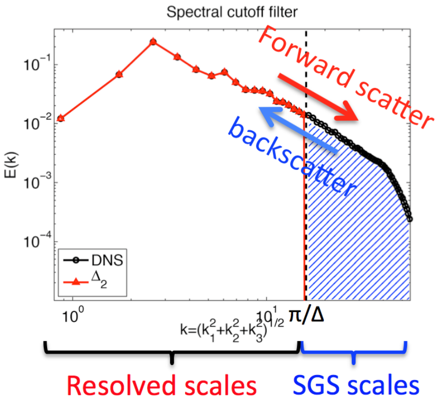
Transfer of energy between resolved and SFSs

- The SFS dissipation Π in the resolved kinetic energy equation is a sink of resolved kinetic energy (it is a source in the k_r equation) and represents the transfer of energy from resolved SFSs. It is equal to:

$$\Pi = -\tau_{ij}\tilde{S}_{ij}$$

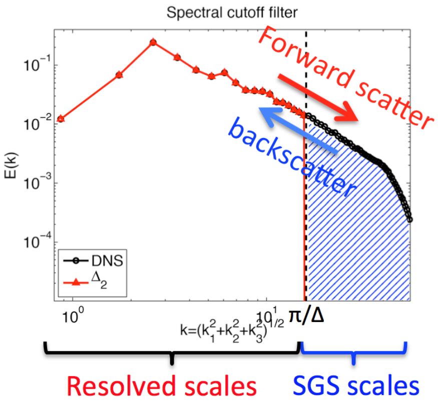
- It is referred to as the SFS dissipation as an analogy to viscous dissipation (and in the inertial subrange $\Pi =$ viscous dissipation).

Transfer of energy between resolved and SFSs



- *On average*, Π drains energy (transfers energy down to smaller scales) from the resolved scales.
- Instantaneously (locally) Π can be positive or negative.

Transfer of energy between resolved and SFSs



- When Π is negative (transfer from SFS $\Rightarrow$ resolved scales) it is typically termed backscatter
- When Π is positive (transfer from resolved scales $\Rightarrow$ SFS) it is sometimes referred to as forward scatter.

Transfer of energy between resolved and SFSs

- It is informative to compare our resolved kinetic energy equation to the mean kinetic energy equation (derived in a similar manner, see Pope pg. 124; Stull 1988 ch. 5)

$$\frac{\partial \langle E \rangle}{\partial t} + \langle u_i \rangle \frac{\partial \langle E \rangle}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \langle u_i \rangle \langle p \rangle}{\partial x_j} - \frac{\partial}{\partial x_j} 2\nu \langle u_i \rangle \langle S_{ij} \rangle - P - \langle \epsilon \rangle$$

where P is shear production and $\langle \epsilon \rangle$ is mean dissipation.

$$P = \langle u_i' u_j' \rangle \frac{\partial \langle u_i \rangle}{\partial x_j}$$

$$\langle \epsilon \rangle = 2\nu \langle S_{ij} \rangle \langle S_{ij} \rangle$$

Transfer of energy between resolved and SFSs

$$\frac{\partial \tilde{E}_f}{\partial t} + \tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j} = -\frac{1}{\rho}\frac{\partial(\tilde{u}_i \tilde{p})}{\partial x_i} - \frac{\partial(\tilde{u}_i \tau_{ij})}{\partial x_j} - 2\nu \frac{\partial(\tilde{u}_i \tilde{S}_{ij})}{\partial x_j} - \epsilon_f - \Pi$$

$$\frac{\partial \langle E \rangle}{\partial t} + \langle u_i \rangle \frac{\partial \langle E \rangle}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \langle u_i \rangle \langle p \rangle}{\partial x_j} - P - 2\nu \frac{\partial}{\partial x_j}\langle u_i \rangle \langle S_{ij} \rangle - \langle \epsilon \rangle$$

- For high-Re flow, with our filter in the inertial subrange: $\langle \tilde{E}_f \rangle = \langle E \rangle$
- The dominant sink for $\langle \tilde{E}_f \rangle$ is $\langle \Pi \rangle$ while for $\langle E \rangle$ it is $\langle \epsilon \rangle$ (rate of dissipation of energy). For high-Re flow we therefore have:

$$\langle \Pi \rangle \approx \langle \epsilon \rangle$$

Transfer of energy between resolved and SFSs

$$\frac{\partial \tilde{E}_f}{\partial t} + \tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j} = -\frac{1}{\rho}\frac{\partial(\tilde{u}_i\tilde{p})}{\partial x_i} - \frac{\partial(\tilde{u}_i\tau_{ij})}{\partial x_j} - 2\nu\frac{\partial(\tilde{u}_i\tilde{S}_{ij})}{\partial x_j} - \epsilon_f - \Pi$$

$$\frac{\partial\langle E\rangle}{\partial t} + \langle u_i\rangle\frac{\partial\langle E\rangle}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\langle u_i\rangle\langle p\rangle}{\partial x_j} - P - 2\nu\frac{\partial}{\partial x_j}\langle u_i\rangle\langle S_{ij}\rangle - \langle\epsilon\rangle$$

- Recall from K41 that $\langle\epsilon\rangle$ is proportional to the transfer of energy in the inertial subrange. Thus, Π will have a strong impact on energy transfer and the shape of the energy spectrum in LES.
- Calculating the correct average Π is another necessary (but not sufficient) condition for an LES SFS model (to go with our N-S invariance properties).