LES of Turbulent Flows: Lecture 2

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Overview

1. Basic Properties of Turbulence
2. Random Nature of Turbulence
3. Statistical Tools for Turbulent Flow

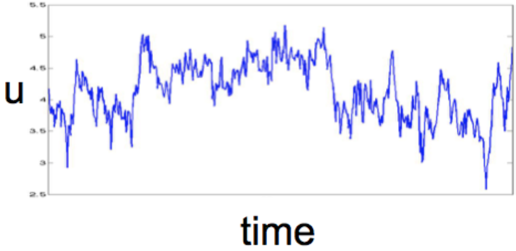
Basic Properties of Turbulence

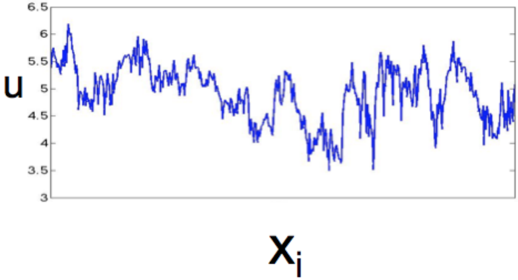
- **Turbulence is random**
  The properties of the fluid (ρ, P, u) at any given point (x,t) cannot be predicted. But statistical properties – time and space averages, correlation functions, and probability density functions – show regular behavior. The fluid motion is stochastic.
- **Turbulence decays without energy input**
  Turbulence must be driven or else it decays, returning the fluid to a laminar state.

Basic Properties of Turbulence

- **Turbulence displays scale-free behavior**
  On all length scales larger than the viscous dissipation scale but smaller than the scale on which the turbulence is being driven, the appearance of a fully developed turbulent flow is the same.
- **Turbulence displays intermittency**
  "Outlier" fluctuations occur more often than chance would predict.

Basic Properties of Turbulent Flows





- Unsteady
- Three-dimensional

Basic Properties of Turbulent Flows

- Large vorticity

- Vorticity describes the tendency of something to rotate.

$$\begin{aligned}\omega &= \nabla \times \vec{u} \\ &= \epsilon_{ijk} \frac{\partial}{\partial x_i} u_j \hat{e}_k \\ &= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \hat{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \hat{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x2} \right) \hat{e}_3\end{aligned}$$

Vortex stretching can and does create small scale circulations that increases the turbulence intensity I, where:

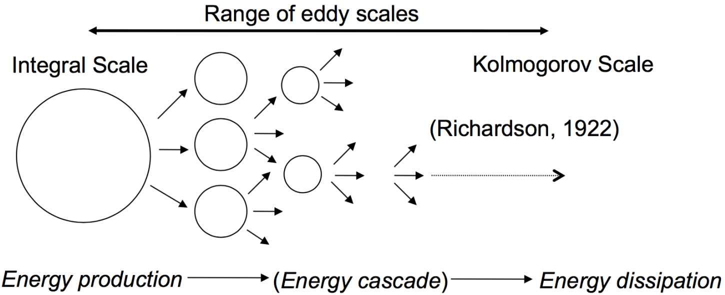
$$I = \frac{\sigma_u}{\langle u \rangle}$$

Basic Properties of Turbulent Flows

- Mixing effect

  Turbulence mixes quantities (*e.g.*, pollutants, chemicals, velocity components, etc)., which acts reduce gradients. This lowers the concentration of harmful scalars, but increases drag.

- A continuous spectrum (range) of scales.



Random Nature of Turbulence

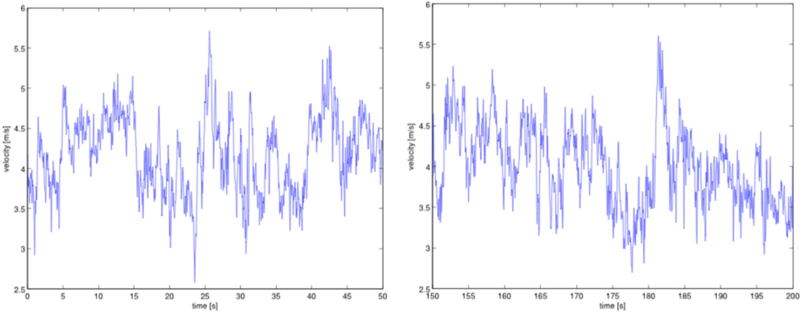
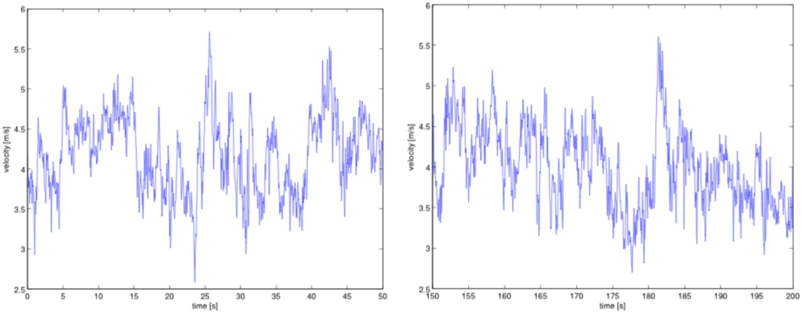


Figure: Sonic anemometer data at 20Hz taken in the ABL.

This velocity field exemplifies the random nature of turbulent flows.

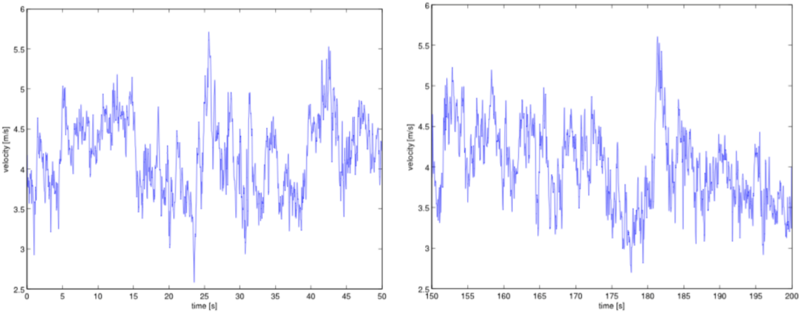
Random Nature of Turbulence



- The signal is highly disorganized and has structure on a wide range of scales (that is also disorganized).

  Notice the small (fast) changes verse the longer timescale changes that appear in no certain order.

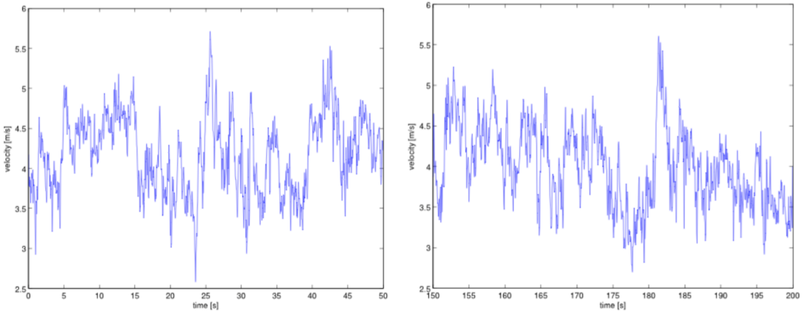
Random Nature of Turbulence



- The signal appears unpredictable.

  Compare the left plot with that on the right (100 s later). Basic aspects are the same but the details are completely different. From looking at the left signal, it is impossible to predict the right signal.

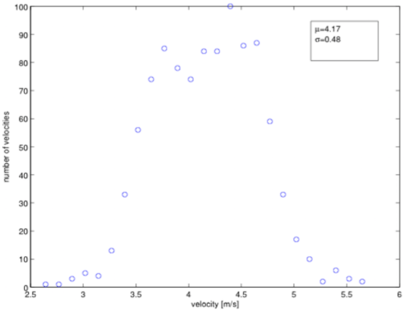
Random Nature of Turbulence

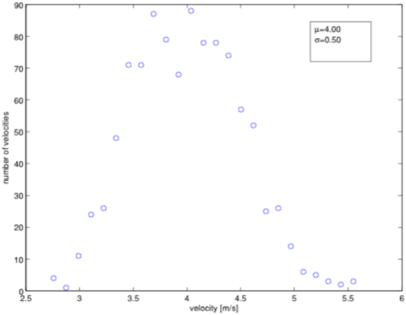


- Some of the properties of the signal appear to be reproducible.

  The reproducible property isn't as obvious from the signal. Instead we need to look at the histogram.

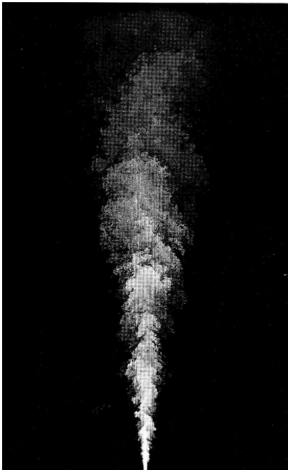
Random Nature of Turbulence

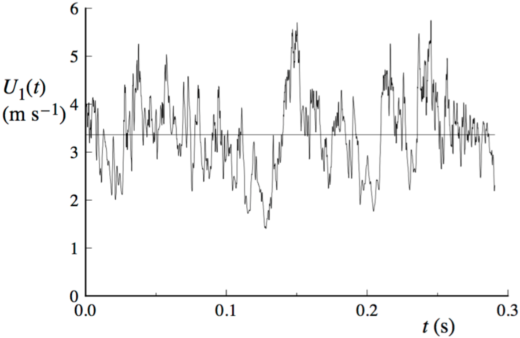




Notice that the histograms are similar with similar means and standard deviations.

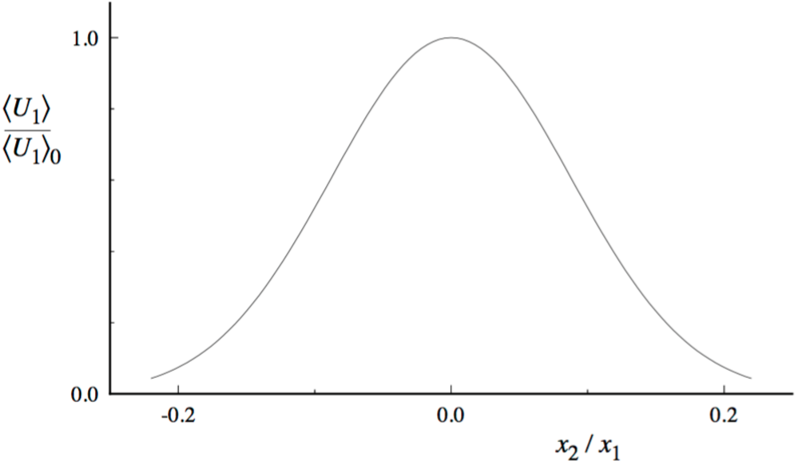
Random Nature of Turbulence





The left panel shows concentration in a turbulent jet, while the right shows the time history along the centerline (see Pope).

Random Nature of Turbulence



Normalized mean axial velocity in a turbulent jet (see Pope).

Random Nature of Turbulence

- The random behavior observed in the time series can appear to contradict what we know about fluids from classical mechanics.
- The Navier-Stokes equations are deterministic (*i.e.*, they give us an exact mathematical description of the evolution of a Newtonian fluid).
- Yet, as we have seen, turbulent flows are random.
- How do we resolve this inconsistency?

Random Nature of Turbulence

Question: Why the randomness?

- There are unavoidable perturbations (*e.g.*, initial conditions, boundary conditions, material properties, forcing, etc.) in turbulent flows.
- Turbulent flows and the Navier-Stokes equations are acutely sensitive to these perturbations.
- These perturbations do not fully explain the random nature of turbulence, since such small changes are present in laminar flows.
- However, the sensitivity of the flow field to these perturbations at large Re is much higher.

Random Nature of Turbulence

- This sensitivity to initial conditions has been explored extensively from the viewpoint of dynamical system. This is often referred to as chaos theory.
- The first work in this area was carried out by Lorenz (1963) in the areas of atmospheric turbulence and predictability. Perhaps you have heard the colloquial phrase, *the butterfly effect*.
- Lorenz studied a system with three state variables x, y, and z (see his paper or Pope for details). He ran one experiment with $x(0) = 1$ and another with $x = 1.000001$, while y and z were held constant.

Random Nature of Turbulence

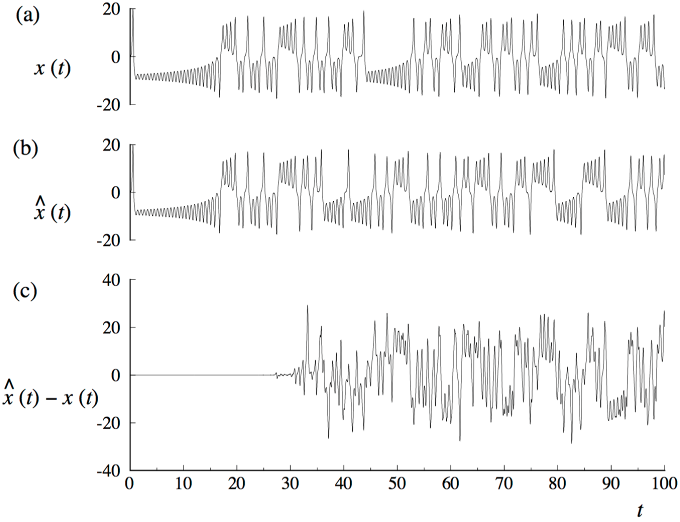


Figure: Time history of the Lorenz equations.

Random Nature of Turbulence

- The work by Lorenz demonstrates the extreme sensitivity to initial conditions.
- The result of this sensitivity is that beyond some point, the state of the system cannot be predicted (*i.e.*, the limits of predictability).
- In the Lorenz example, even when the initial state is known to within 10^{-6}, predictability is limited to $t = 35$.

Random Nature of Turbulence

- In the Lorenz example, this behavior depends on the coefficients of the system. If a particular coefficient is less than some critical value, the solutions are stable. If, on the other hand, it exceeds that value, then the system becomes chaotic.
- This is similar to the Navier-Stokes equations, where solutions are steady for a sufficiently small Re, but turbulent if Re becomes large enough.

Random Nature of Turbulence

- We have seen that turbulent flows are random, but their histograms are apparently reproducible.
- As a consequence, turbulence is usually studied from a statistical viewpoint.

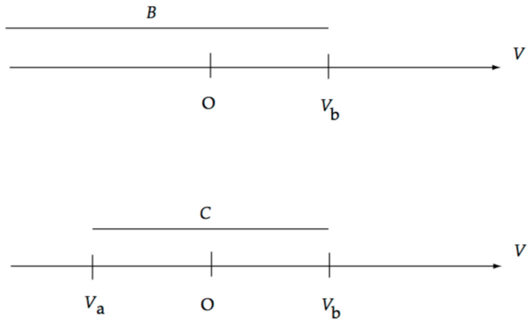
Basic Properties of Turbulence

- Consider the velocity field U.
- Since U is a random variable, its value is unpredictable for a turbulent flow.
- Thus, any theory used to predict a particular value for U will likely fail.
- Instead, theories should aim at determining the probability of events (*e.g.*, $U < 10 \mathrm{~m~s^{-1}}$).
- We need statistical tools to characterize random variables.

Sample Space

- In reality, a velocity field $U(\vec{x}, t)$ is more complicated than a single random variable.
- In order to consider more general events than our example of $U < 10 \text{ m s}^{-1}$, we need to think in terms of *sample space*.
- Consider an independent velocity variable V, which is the sample-space variable for U.

Sample Space



$$B \equiv \{U < V_b\}$$
$$C \equiv \{V_a \leq U < V_b\}$$

B and C are events (or values) that correspond to different regions of the sample space (*i.e.*, velocity field).

Probability

- Using the previous example, the probability of event B is given as:

$$p = P(B) = P\{U < V_b\}$$

- This is the likelihood of B occurring ($U < V_b$).
- p is a real number, $0 \leq p \leq 1$.
- $p = 0$ is an impossible event.
- $p = 1$ is a certain event.

Cumulative Distribution Function

- The probability of any event is determined by the cumulative distribution function (CDF)

$$F(V) \equiv P\{U < V\}$$

- For event B:

$$P(B) = \{U < V_b\} = F(V_b)$$

- For event C:

$$\begin{aligned} P(C) &= \{V_a \leq U < V_b\} = P\{U < V_b\} - P\{U < V_a\} \\ &= F(V_b) - F(V_a) \end{aligned}$$

Cumulative Distribution Function

Three basic properties of CDF:

- $F(-\infty) = 0$, since $\{U < -\infty\}$ is impossible.
- $F(\infty) = 1$, since $\{U > \infty\}$ is impossible.
- $F(V_b) \geq F(V_a)$, for $V_b > V_a$, since $p > 0$. Thus, F is a non-decreasing function.

Probability Density Function

The probability density function (PDF) is the derivative of the CDF

$$f(V) \equiv \frac{dF(v)}{dV}$$

Based on the properties of the CDF, it follows that:

- $f(V) \geq 0$
- $\int_{-\infty}^{\infty} f(V) dV = 1$
- $f(-\infty) = f(\infty) = 0$

Probability Density Function

The probability that a random variable is contained within a specific interval is the integral of the PDF over that interval

$$P(C) = P\{V_a \leq U < V_b\} = F(V_b) - F(V_a)$$
$$= \int_{V_a}^{V_b} f(V)dV$$

Or, for a very small interval dV_s:

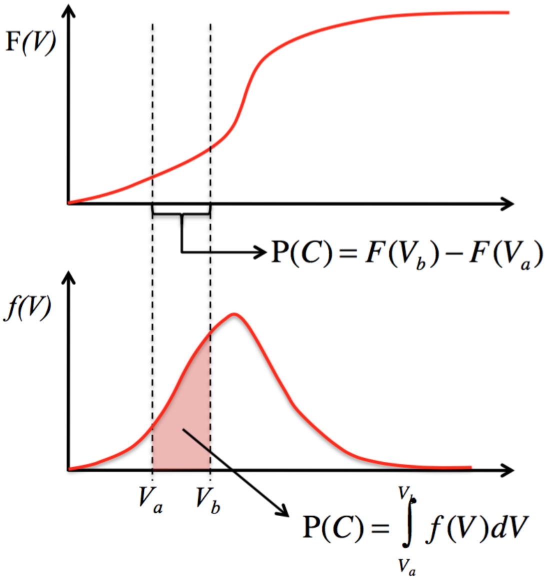
$$P(C_s) = P\{V_a \leq U < V_a + dV_s\} = F(V_a + dV_s) - F(V_a)$$
$$= f(V_a)dV_s$$

Probability Density Function

More details about the PDF $f(V)$:

- $f(V)$ is the probability per unit distance in the sample space – hence, the term *density*.
- $f(V)$ has dimensions of U^{-1}, while the CDF is dimensionless.
- The PDF fully characterizes the statistics of a signal (random variable).
- If two or more signals have the same PDF, then they are considered to be statistically identical.

CDF (top) vs. PDF (bottom)



Means and Moments

We can also define a signal by its individual statistics, which collectively describe the PDF.

The mean (or expected) value of a random variable U is given by:

$$\langle U \rangle \equiv \int_{-\infty}^{\infty} V f(V) dV$$

or in discrete form:

$$\langle U \rangle \equiv \frac{1}{N} \sum_{i=1}^{N} V_i$$

The mean represents the probability-weighted sum of all possible values of U.

Means

Consider some function of U, $Q(U)$.

$$\langle Q(U)\rangle \equiv \int_{-\infty}^{\infty} Q(V)f(V)dV$$

The mean $\langle Q(U)\rangle$ only exists if the above integral converges absolutely. From this equation, we can show that for $Q(U)$, $R(U)$, and constants a and b:

$$\langle aQ(U) + bR(U)\rangle = a\langle Q(U)\rangle + b\langle R(U)\rangle$$

Thus, $\langle\ \rangle$ behave as a linear operator.

Means

Although U, $Q(U)$, and $R(U)$ are random variables, $\langle U \rangle$, $\langle Q(U) \rangle$, and $\langle R(U) \rangle$ are not.

Thus, the mean of the mean is just the mean (*i.e.*, $\langle\langle U \rangle\rangle = \langle U \rangle$).

Variance

Let's define a *fluctuation* (or perturbation) from the mean:

$$u' \equiv U - \langle U \rangle$$

The *variance* is just the mean-square fluctuation:

$$\begin{aligned} \sigma_u^2 = \mathsf{var}(U) &= \langle u'^2 \rangle \\ &= \int_{-\infty}^{\infty} (V - \langle U \rangle)^2 f(V) dV \end{aligned}$$

Or, in discrete form:

$$\sigma_u^2 = \frac{1}{N - 1^*} \sum_{i=1}^{N} (V_i - \langle U \rangle)^2$$

Variance essentially measures how far a set of (random) numbers are spread out from their mean.

*note the $(N-1)$. This is the Bessel correction – used to correct for bias.

Standard Deviation

The *standard deviation*, or root-mean square (rms) deviation, is just the square-root of the variance:

$$\sigma_u \equiv \mathsf{sdev}(U) = \sqrt{\sigma_u^2} = \langle {u'}^2 \rangle^{0.5}$$

The standard deviation basically measures the amount of variation of a set of numbers.

Other moments

The n^{th} central moment is defined as:

$$\mu_n \equiv \langle u'^n \rangle = \int_{-\infty}^{\infty} (V - \langle U \rangle)^n f(V) dV$$

Standardized Moments

It is often advantageous to express variables as standardized random variables. These standardized variables have zero mean and unit variance.

The standardized version of U (centered and scaled) is given by:

$$\hat{U} \equiv \frac{U - \langle U \rangle}{\sigma_u}$$

Accordingly, the n^{th} standardized moments are expressed as:

$$\hat{\mu}_n \equiv \frac{\langle u'^n \rangle}{\sigma_u^n} = \frac{\mu_n}{\sigma_u^n} = \int_{-\infty}^{\infty} \hat{V}^n \hat{f}(\hat{V}) d\hat{V}$$

Other moments

Different moments each describe an aspect of the shape of the PDF:

- μ_1 = mean (expected value)
- μ_2 = variance (spread from the mean)
- $\hat{\mu}_3$ = skewness (asymmetry of PDF)
- $\hat{\mu}_4$ = kurtosis (sharpness of the PDF peak)

Example PDFs

Read Pope (Chapter 3.3) for descriptions of different PDFs.

Examples include:

- uniform
- exponential
- Gaussian
- log-normal
- gamma
- Delta-function
- Cauchy

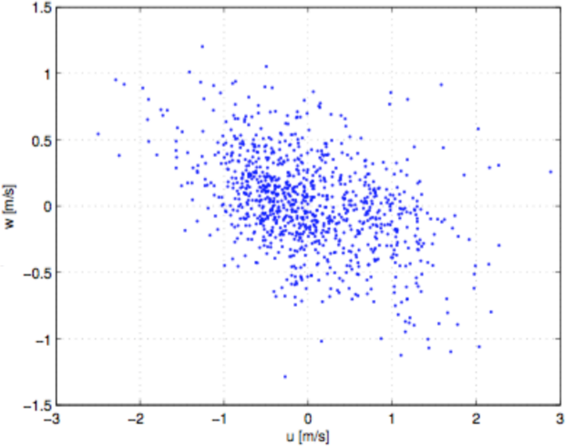
Joint Random Variables

So far, our statistical description has been limited to single random variables. However, turbulence is governed by the Navier-Stokes equations, which are a set of 3 coupled PDEs.

We expect this will result in some correlation between different velocity components.

Joint Random Variables

Example: turbulence data from the ABL: scatter plot of horizontal (u) and vertical (w) velocity fluctuations.



The plot appears to have a pattern (*i.e.*, negative slope).

Joint Random Variables - Sample Space

We will now extend the previous results from a single velocity component to two or more.

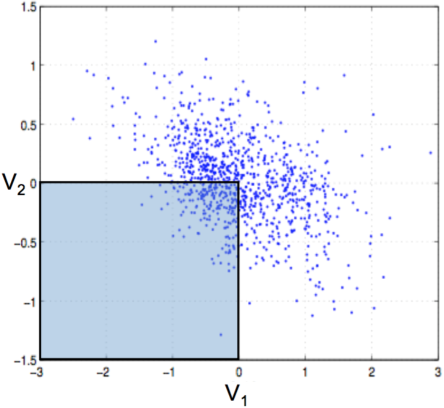
The sample-space variables corresponding to the random variables $U = \{U_1, U_2, U_3\}$ are given by $V = \{V_1, V_2, V_3\}$.

Joint Random Variables - CDF

The joint CDF (jCDF) of the random variables (U_1, U_2) is given by:

$$F_{12}(V_1, V_2) \equiv P\{U_1 < V_1, U_2 < V_2\}$$

This is the probability of the point (V_1, V_2) lying inside the shaded region



The jCDF has the following properties:

- $F_{12}(V_1 + \delta V_1, V_2 + \delta V_2) \geq F_{12}(V_1, V_2)$ (non-decreasing) for all $\delta V_1 \geq 0$, $\delta V_2 \geq 0$
- $F_{12}(-\infty, V_2) = P\{U_1 < -\infty, U_2 < V_2\} = 0$ since $\{U_1 < -\infty\}$ is impossible.
- $F_{12}(\infty, V_2) = P\{U_1 < \infty, U_2 < V_2\} = P\{U_2 < V_2\}$ $F_{12}(\infty, V_2) = F_2(V_2)$, since $\{U_1 < \infty\}$ is certain.

In the last example, $F_2(V_2)$ is called the *marginal CDF*.

Joint Random Variables - PDF

The joint PDF (jPDF) is defined as:

$$f_{12}(V_1, V_2) \equiv \frac{\partial^2}{\partial V_1 \partial V_2} F_{12}(V_1, V_2)$$

If we integrate over V_1 and V_2, we get the probability:

$$P\{V_{1a} \leq U_1 < V_{1b}, V_{2a} \leq U_2 < V_{2b}\} = \int_{V_{1a}}^{V_{1b}} \int_{V_{2a}}^{V_{2b}} f_{12}(V_1, V_2) dV_2 dV_1$$

Joint Random Variables - PDF

Based on the jCDF, the jPDF has the following properties:

- $f_{12}(V_1, V_2) \geq 0$
- $\int_{-\infty}^{\infty} f_{12}(V_1, V_2) dV_1 = f_2(V_2)$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{12}(V_1, V_2) dV_1 dV_2 = 1$

In the middle example, $f_2(V_2)$ is called the *marginal PDF*.
Practically speaking, we find the PDF of a time (or space) series by:

- Create a histogram of the series(group values into bins)
- Normalize the bin weights by the total # of points

Joint Random Variables - Means

Similar to the single variable form, if we have $Q(U_1, U_2)$:

$$\langle Q(U_1, U_2) \rangle \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(V_1, V_2) f_{12} dV_2 dV_1$$

We can use this equation to define a few important statistics.

Joint Random Variables - Covariance

We can define *covariance* as:

$$\text{cov}(U_1, U_2) \equiv \langle u_1' u_2' \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (V_1 - \langle U_1 \rangle)(V_2 - \langle U_2 \rangle) f_{12}(V_1, V_2) dV_2 dV_1$$

Or for discrete data

$$\text{cov}(U_1, U_2) \equiv \langle u_1' u_2' \rangle = \frac{1}{N-1} \sum_{j-1}^{N} (V_{1j} - \langle U_1 \rangle)(V_{2j} - \langle U_2 \rangle)$$

Covariance is basically a measure of how much two random variables change together.

Joint Random Variables - Correlation Coefficient

The *correlation coefficient* is given by:

$$\rho_{12} \equiv \frac{\langle u_1' u_2' \rangle}{\sqrt{\langle {u_1'}^2 \rangle \langle {u_2'}^2 \rangle}}$$

Correlation coefficient has the following properties:

- $-1 \leq \rho_{12} \leq 1$
- Positive values indicate correlation.
- Negative values indicate anti-correlation.
- $\rho_{12} = 1$ is perfect correlation.
- $\rho_{12} = -1$ is perfect anti-correlation.

Two-point Statistical Measures

Autocovariance measures how a variable changes with different lags, s.

$$R(s) \equiv \langle u(t)u(t+s) \rangle$$

or the *autocorrelation function*

$$\rho(s) \equiv \frac{\langle u(t)u(t+s) \rangle}{u(t)^2}$$

Or for the discrete form

$$\rho(s_j) \equiv \frac{\sum_{k=0}^{N-j-1} (u_k u_{k+j})}{\sum_{k=0}^{N-1} (u_k^2)}$$

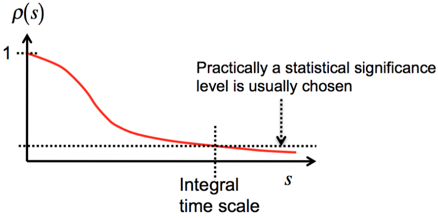
Two-point Statistical Measures

Notes on autocovariance and autocorrelation

- These are very similar to the covariance and correlation coefficient
- The difference is that we are now looking at the linear correlation of a signal with itself but at two different times (or spatial points), i.e. we lag the series.
- We could also look at the cross correlations in the same manner (between two different variables with a lag).
- $\rho(0) = 1$ and $|\rho(s)| \leq 1$

Two-point Statistical Measures

- In turbulent flows, we expect the correlation to diminish with increasing time (or distance) between points
- We can use this to define an integral time (or space) scale. It is defined as the time lag where the integral $\int \rho(s)ds$ converges.
- It can also be used to define the largest scales of motion (statistically).



Two-point Statistical Measures

The *structure function* is another important two-point statistic.

$$D_n(r) \equiv \langle [U_1(x+r,t) - U_1(x,t)]^n \rangle$$

- This gives us the average difference between two points separated by a distance r raised to a power n.
- In some sense it is a measure of the moments of the velocity increment PDF.
- Note the difference between this and the autocorrelation which is statistical linear correlation (*i.e.*, multiplication) of the two points.