Environmental Fluid Dynamics: Lecture 24

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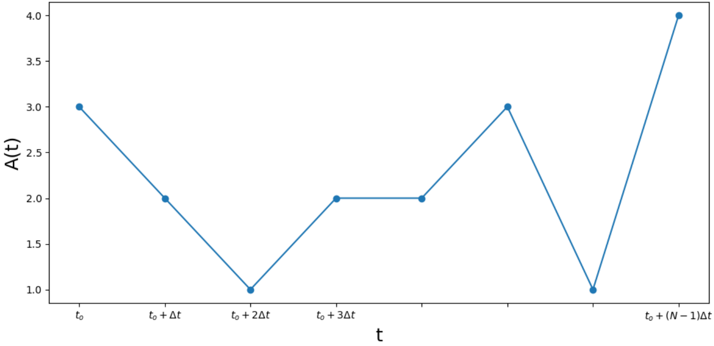
Overview

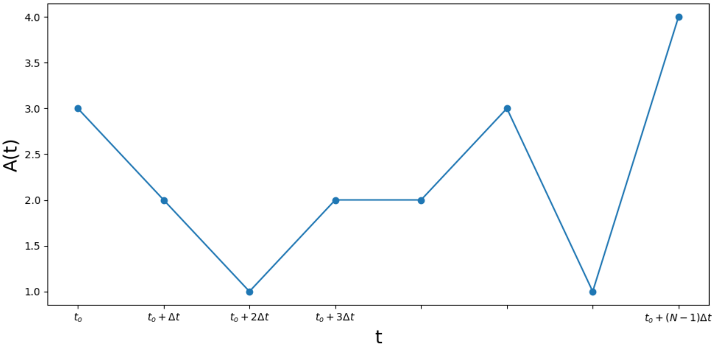
Discrete Time Series Analysis

Discrete Time (or Space) Series

- Consider uniformly spaced data in time (could be space).
- We will call this series discrete because there is a finite number of data points, which represent a sample of the true continuously-varying signal.



Discrete Time (or Space) Series



- $N \rightarrow$ total number of data points
- $t_o + k\Delta t \rightarrow k^{th}$ data point $(0 \leq k \leq N-1)$
- $A(t_k) = A(k) = A_k \rightarrow$ notation for sampled series
- $T = N\Delta t \rightarrow$ total period of sampling

Taylor's Frozen Turbulence Hypothesis

> *If the velocity of the air stream which carries the eddies is very much greater than the turbulent velocity, one may assume that the sequence of changes in u at the fixed point are simply due to the passage of an unchanging pattern of turbulent motion over the point, i.e. one may assume that*
>
> $$u = \phi(t) = \phi\left(\frac{x}{U}\right)$$
>
> *where x is measured at time $t = 0$ from the fixed point where u is measured.*

Taylor's Frozen Turbulence Hypothesis

- As a result turbulence measurements that are made as a function of time can be translated into a corresponding spatial measurement.
- This hypothesis is useful for cases where turbulent eddies evolve with a timescale longer than the time scale it takes the eddy to be advected past the sensor.

Taylor's Frozen Turbulence Hypothesis

- Following Stull (1988), the substantial derivative is zero for Taylor's Hypothesis
- Thus,

$$\frac{\partial \zeta}{\partial t} = -\overline{u}\frac{\partial \zeta}{\partial x} - \overline{v}\frac{\partial \zeta}{\partial y} - \overline{w}\frac{\partial \zeta}{\partial z}$$

- If we assume that $\overline{w} = 0$ and write $U = \sqrt{\overline{u}^2 + \overline{v}^2}$, then

$$\frac{\partial \zeta}{\partial t} = -U\frac{\partial \zeta}{\partial x_d}$$

where x_d indicates along the direction of the wind.

Taylor's Frozen Turbulence Hypothesis

- We can also write Taylor's hypothesis in terms of wavenumber k and frequency f:

$$k = \frac{f}{U}$$

  where $k = 2\pi/\lambda$ and $f = 2\pi/T$ for wavelength λ and wave period T.
- k has dimensions of radians per unit length.
- f has dimensions of radians per unit time.

Estimating Dissipation Rate

- Recall when we derived the turbulence kinetic energy balance equation that dissipation was written as:

$$\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}$$

- Assuming homogeneous isotropic turbulence, dissipation may be estimated as:

$$\epsilon = 15\nu \overline{\left(\frac{\partial u'}{\partial x} \right)^2}$$

- We can invoke Taylor's frozen turbulence hypothesis to rewrite as

$$\epsilon = 15\nu \overline{\left(-\frac{1}{U} \frac{\partial u'}{\partial t} \right)^2}$$

where $\partial u' / \partial t$ is approximated, for example, from measurements.

Estimating Dissipation Rate

$$\epsilon = 15\nu\overline{\left(-\frac{1}{U}\frac{\partial u'}{\partial t}\right)^2}$$

- Remember that dissipation occurs at very small time and space scales.
- Thus, our measurement probes must be small and sample at high frequencies.
- Examples are sonic anemometers or hot-wire probes.

Autocorrelation

- Consider the discrete autocorrelation, which measures the persistence of a wave within the duration of a discrete series.
- Existence of persistent features may point to particular physical phenomena (e.g., eddy).

$$R_{AA}(L) = \frac{\displaystyle\sum_{k=0}^{N-j-1} \left[\left(A_k - \overline{A}_k\right)\left(A_{k+j} - \overline{A}_{k+j}\right)\right]}{\left[\displaystyle\sum_{k=0}^{N-j-1} \left(A_k - \overline{A}_k\right)^2\right]^{1/2} \left[\displaystyle\sum_{k=0}^{N-j-1} \left(A_{k+j} - \overline{A}_{k+j}\right)^2\right]^{1/2}}$$

where the lag $L = j\Delta t$

Autocorrelation

- Note that we use two different means depending on where we are in the time series

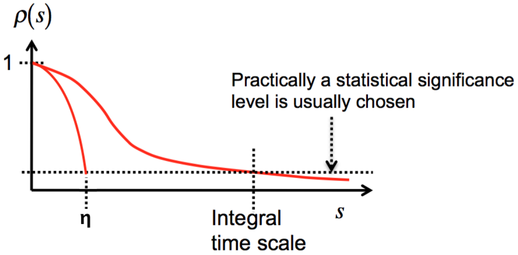
$$\overline{A}_k = \frac{1}{N-j} \sum_{k=0}^{N-j-1} A_k \qquad \overline{A}_{k+j} = \frac{1}{N-j} \sum_{k=0}^{N-j-1} A_{k+j}$$

- If we assume that the data is stationary (homogeneous in space):

$$R_{AA}(L) \simeq \frac{\overline{A'_k A'_{k+j}}}{\sigma_A^2}$$

- As lag increases, we use less of the series and so the statistical significance of R_{AA} decreases.
- Thus, we compute R_{AA} for the range of lags ($j = 0$ to $j = N/2$).

Autocorrelation



- Autocorrelation can aid in showing the persistence of an eddy
- Integral scale $\ell_o = \int_0^\infty R_{AA}(L)dL$ is a measure of the area over which a signal is correlated with itself (indicates largest eddies in the flow).
- Kolmogorov microscale is found by fitting a parabola to the near-origin points (see Tennekes and Lumley) and locating the x-intercept. This measures the smallest eddies that are dynamically significant in the flow.

Discrete Fourier Transform

There are several ways to describe the frequency of our series.

- $n =$ number of cycles per time period (from 1 to $N - 1$)
- $\tilde{n} =$ cycles per second: $n/T = n/(N\Delta t)$
- $f =$ radians per second: $2\pi n/T = 2\pi n/(N\Delta t)$

Frequency values mean different things

- $n = 0 \rightarrow$ mean value
- $n = 1 \rightarrow$ fundamental frequency (one wave fills T)
- $n > 1 \rightarrow$ harmonics of the fundamental frequency

Discrete Fourier Transform

- We can represent our series as the superposition of sine and cosine waves via Euler's formula $[\exp(ix) = \cos(x) + i\sin(x)]$

$$A_k = \sum_{k=0}^{N-1} F_A(n) e^{i2\pi nk/N}$$

  where $F_A(n)$ is the **discrete Fourier transform**.
- $F_A(n)$ is a complex number where the real part is the amplitude of the cosine waves and the imaginary part is the amplitude of the sine waves.
- $F_A(n)$ is a function of frequency because waves of different frequencies have to be multiplied by different amplitudes to reconstruct the signal.

Discrete Fourier Transform

- If we have the discrete series, we can solve for the Fourier coefficients

$$F_A(n) = \sum_{k=0}^{N-1} \left(\frac{A_k}{N} \right) e^{-i2\pi nk/N}$$

- This is the forward transform, which converts from physical to phase space.
- Another name for this expression is **Fourier decomposition**.

Discrete Energy Spectrum

- We are interested in how much variance of a discrete series is associated with a particular frequency.
- We are not interested in the phase of the waves
- In fact, we expect that a turbulent signal does not behave physically like a wave at all.
- It is still useful to break the turbulent signal into components of different frequencies, which we associate with eddies of different sizes.
- i.e., large eddies have low frequency and small eddies have high frequency

Discrete Energy Spectrum

- Note that the signal power (power at different frequencies) is defined as:

$$P = \frac{1}{T} \int A^2(t)dt$$

- So we need to find the square of the norm of our transform

$$\begin{aligned}
F_A(n) &= \underbrace{F_r}_{\text{real}} + \underbrace{iF_i}_{\text{imag}} \\
F_A^*(n) &= \underbrace{F_r - iF_i}_{\text{complex conjugate}} \\
|F_A(n)|^2 &= F_A(n)F_A^*(n) \\
&= (F_r + iF_i)(F_r - iF_i) \\
&= F_r^2 - F_r iF_i + F_r iF_i + F_i^2 \\
&= F_r^2 + F_i^2
\end{aligned}$$

Discrete Energy Spectrum

- In Matlab:

$$|F_u(n)|^2 = \left[\frac{\mathsf{fft}(u)}{\mathsf{length}(u)}\right] .* \,\mathsf{conj}\left[\frac{\mathsf{fft}(u)}{\mathsf{length}(u)}\right]$$

- If we sum up $|F_A(n)|^2$ from $n = 1$ to $N - 1$, we get the total biased variance

$$\sum_{n=1}^{N-1} |F_A(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1}(A_k - \overline{A}_k)^2 = \sigma_A^2$$

- Thus, we say that the $|F_A(n)|^2$ is the portion of variance explained by waves of frequency n.
- Note: we don't sum over $n = 0$ because that represents the mean value of the signal, which does not contribute to the variation of the signal about the mean.

Discrete Energy Spectrum

- If we define $G_A(n) = |F_A(n)|^2$, then:

$$\frac{G_A(n)}{\sigma_A^2}$$

describes the fraction of the variance explained by frequency n. In this sense, it is analogous to the correlation coefficient.

- We can write the **discrete spectral energy** $E_A(n)$ as:

$$E_A(n) = 2|F_A(n)|^2$$

for $n = 1$ to n_f when N is odd, or

$$E_A(n) = 2|F_A(n)|^2$$

for $n = 1$ to $n_f - 1$ and

$$E_A(n) = |F_A(n)|^2$$

at $n = n_f$ when N is even.

Discrete Energy Spectrum

- The discrete spectral energy may be used for variables such as temperature, humidity, and velocity in order to separate the total variance into contributions by different frequencies.
- Be careful not to assume that spectra of temperature and humidity relate to eddy motions since variations of these variables can persist in a non-turbulent flow as a "footprint" of previous turbulent activity.
- An example is the residual layer that persists after sundown, when the gradients of moisture and temperature can maintain their shapes that were created during the convective boundary layer.
- The variance of velocity fluctuations u' has the same units as turbulence kinetic energy per unit mass - thus, the spectrum of velocity is often called the **energy spectrum**.

Spectral Density

- Several theories use continuous spectra instead of discrete spectra.
- Instead of summing discrete spectra over all n to obtain the total variance, they assume the existence of a **spectral energy density**, that can be integrated over n to yield the total variance:

$$\sigma_A^2 = \int_n S_A(n)dn$$

- The spectral energy density $S_A(n)$ has units of A^2 per unit frequency.
- We can approximate the spectral energy density as:

$$S_A(n) = \frac{E_A(n)}{\Delta n}$$

Spectral Density

$$S_A(n) = \frac{E_A(n)}{\Delta n}$$

- Δn is the difference between neighboring frequencies.
- When n is used to represent frequency, then $\Delta n = 1$. Other representations, such as f, do not necessarily lead to $\Delta n = 1$.
- The $S_A(n)$ points are plotted as curve to represent the spectrum.

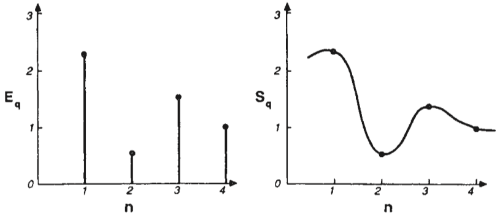


Fig. 8.8 (a) Discrete spectrum and (b) spectral density graphs for example 8.6.3.

via Stull (1988)

Spectral Density

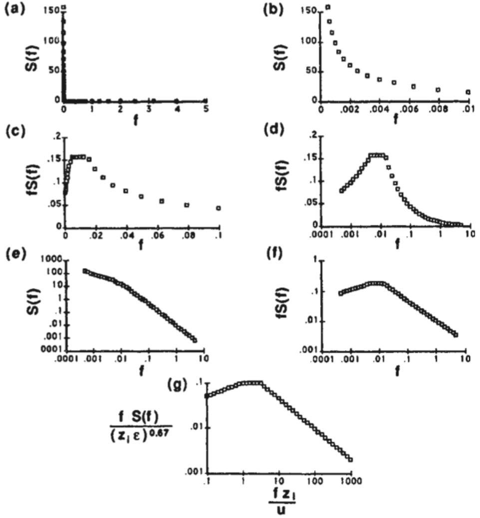


Fig. 8.9 Different presentations of the same spectrum (see text for details).

via Stull (1988)

Spectral Density

linear-linear plots

- As in panel (a), area under curve between pair of frequencies is proportional to the portion of variance explained by that range of frequencies.
- Visually useless because high-frequency scales are masked by the large values at low frequencies.
- Alternative are to expand the low-frequency part of the spectrum (panel b), or to multiply the spectral density by f (panel c).
- These approaches still focus on the spectral peak and lose information at high frequencies.

Spectral Density

semi-log plots

- In this approach (panel d), $f \cdot S_A(f)$ is plotted against $\log(f)$
- Making the x-axis a log scale results in the expansion of the low-frequency parts of the spectrum.
- Multiplying the spectral density by f results in the expansion of the high-frequency parts of the spectrum alongt he y-axis.
- The area under any part of the curve is proportional to the variance.

Spectral Density

log-log plots

- This presentation (panel e) is $\log S_A(f)$ vs. $\log f$.
- A wide range of frequencies and spectral densities are discernible.
- Power laws (such as Kolmogorov's $-5/3$ law) appear as straight lines.
- The area under a curve is no longer proportional to variance.

Spectral Density

log-log plots

- Another version of the log-log plot (panel f) is $\log f \cdot S_A(f)$ vs. $\log f$.
- Same characteristics of previous log-log plot.
- $f \cdot S_A(f)$ has the same units as variance, which makes normalization easier.
- The area under a curve is also no longer proportional to variance.
- One last approach is a normalize (make dimensionless) the x-axis and y-axis by way of scaling variables (panel g).

Cross Spectra

- Using the ideas before for one variable, consider the cross spectra of two variables

$$\begin{aligned} G_{AB} &= F_A^*(n)F_B(n) \\ &= (F_{Ar} - iF_{Ai})(F_{Br} + iF_{Bi}) \\ &= F_{Ar}F_{Br} - iF_{Ai}F_{Br} + F_{Ar}iF_{Bi} + F_{Ai}F_{Bi} \\ &= C_o - iQ \end{aligned}$$

  where the real parts make up the co-spectrum (C_o) and the imaginary parts make up the quadrative (Q) spectrum

$$\begin{aligned} C_o &= F_{Ar}F_{Br} + F_{Ai}F_{Bi} \\ Q &= F_{Ai}F_{Br} - F_{AR}F_{Bi} \end{aligned}$$

Cospectrum

- Like with variance and energy spectrum, the sum over all frequencies of all co-spectral amplitudes is the covariance of A and B.

$$\sum_{n=0}^{N-1} C_o(n) = \overline{A'B'}$$

- This is **not** the same as the spectrum of the time series of $A'B'$
- As a result, the co-spectrum can have negative values.
- Recall, energy spectrum cannot (magnitude).

Discrete Energy Cospectrum

- We can write the **discrete cospectral energy** $E_A(n)$ as:

$$E_{AB}(n) = 2|C_o(n)|^2$$

for $n = 1$ to n_f when N is odd, or

$$E_{AB}(n) = 2|C_o(n)|^2$$

for $n = 1$ to $n_f - 1$ and

$$E_{AB}(n) = |C_o(n)|^2$$

at $n = n_f$ when N is even.

Cospectral Density

- We can approximate the co-spectral energy density as:

$$CS_{AB}(n) = \frac{E_{AB}(n)}{\Delta n}$$

- And

$$\overline{A'B'} = \int_n CS_{AB}(n) dn$$

- As $CS_{AB}(n) \to \infty$ at high frequency, we have an indicator of local isotropy.

Phase Spectrum

- The phase spectrum Φ is defined as:

$$\tan \Phi = \frac{Q}{C_o}$$

- This is interpreted as the phase difference between the two time series A and Bthat yields the greatest correlation for any frequency.
- This helps understand the physical structure of the flow.