Environmental Fluid Dynamics: Lecture 23

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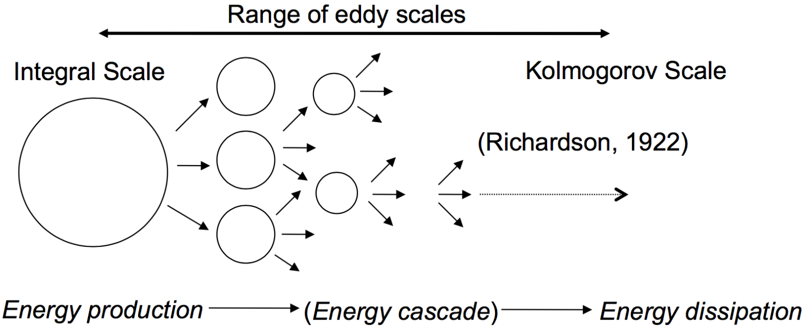
Overview

1. Turbulent Energy Cascade
2. Kolmogorov's similarity hypothesis
3. Fourier transforms

Turbulent Energy Cascade

Turbulent Energy Cascade

Recall that one of the properties of turbulent flows is a continuous spectrum (range) of scales.

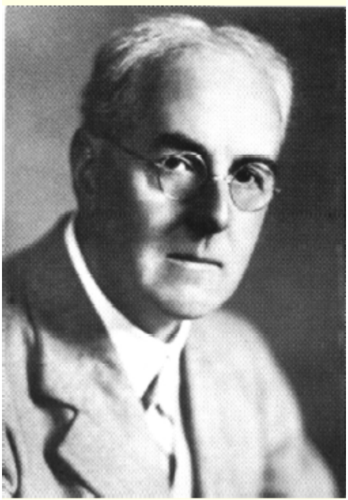


Turbulent Energy Cascade

- Consider a fully developed turbulent flow at high Reynolds number
$$\mathrm{Re} = \frac{\mathcal{U}\mathcal{L}}{\nu}$$
where $\mathcal{U}$ is the characteristic velocity scale of the flow and $\mathcal{L}$ is the characteristic length scale of the flow.
- As we covered earlier in the course, we expect Re to be large for a turbulent flow (larger means a more developed flow).
- Let's apply the *Richardson conceptual model of turbulence* to understand the flow.

Turbulent Energy Cascade

- Lewis Fry Richardson (1881–1953)
- Pioneered the idea of predicting weather by solving differential equations.
- *Weather Prediction by Numerical Process* (1922)



Turbulent Energy Cascade

Richardson, from *Weather Prediction by Numerical Process* (1922)

> *Big whorls have little whorls*
> *That feed on their velocity;*
> *And little whorls have lesser whorls*
> *And so on to viscosity.*

Turbulent Energy Cascade

- Richardson's concept of turbulence implies that the flow is composed of turbulent eddies.
- Consider a generic eddy of size (scale) ℓ.
- This eddy will have a velocity scale $u(\ell)$ and time scale $\tau(\ell) \equiv l/u(\ell)$.
- The eddy is considered to be a unit carrier of turbulent motion.
- Furthermore, the eddy is localized within a region of size ℓ and assumed to be at least moderately coherent over this region.

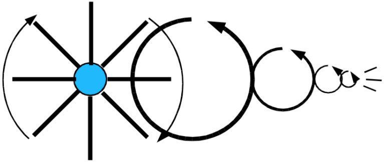
Turbulent Energy Cascade

- If we look at the largest eddies in the flow, they are on the scale of the flow itself.
- We call this largest scale the integral scale ($\ell_o \sim \mathcal{L}$), which is on the order of the auto-correlation length - or in a boundary layer, the integral scale is comparable to the boundary layer depth.
- The associated velocity scale $u_o(\ell_o)$ is on the order of the turbulence intensity.
- The Reynolds number of these eddies, Re$= u_o \ell_o / \nu$, is large relative to that of smaller eddies in the flow and is comparable to Re.

Turbulent Energy Cascade

The idea of the turbulent cascade:

- Vorticity is created on large scales by some driving mechanism that feeds energy to the fluid, which results in these large eddies.
- Shear instability causes smaller vortices to be shed, drawing energy from the larger ones.
- This process continues on ever smaller scales.
- On the smallest scales, diffusion destroys eddies and converts their kinetic energy to thermal energy.

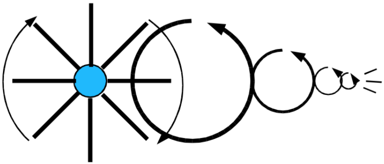


Turbulent Energy Cascade

The idea of the turbulent cascade:

- The cascade is considered a closed system.
- The energy transfer rate (energy per unit mass per unit time) by large eddies at the start (s) of the cascade is equal to the energy transfer rate at the finish (f) of the cascade:

$$P_s \sim \frac{u_s^2}{\tau_s} \sim \frac{u_s^2}{\ell_s / u_s} \sim \frac{u_s^3}{\ell_s} \simeq \frac{u_f^3}{\ell_f} \simeq P_f$$



Turbulent Energy Cascade

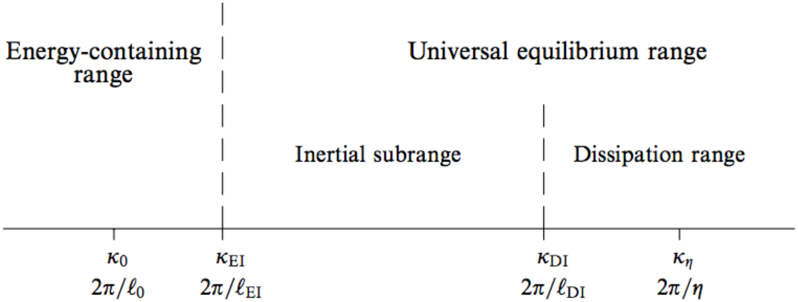
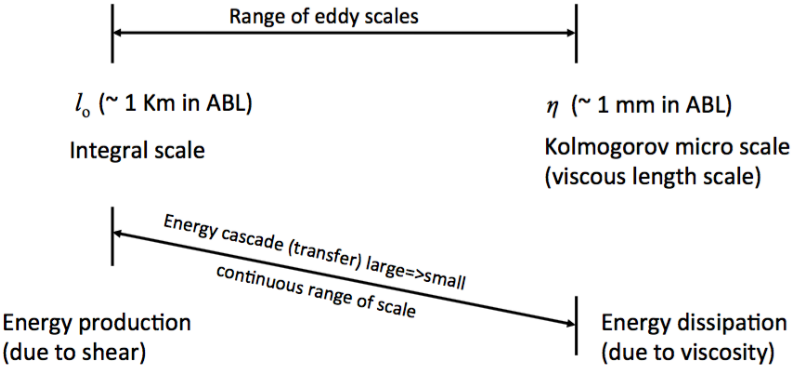
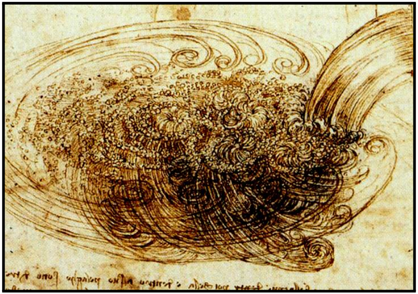


Fig. 6.12. Wavenumbers (on a logarithmic scale) at very high Reynolds number showing the various ranges.

Turbulent Energy Cascade



Remember da Vinci?



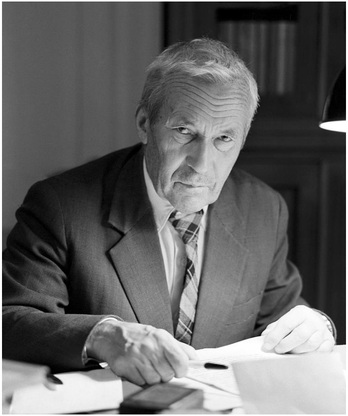
> *… the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large.*

Sounds like Richardson's turbulent cascade!

Kolmogorov's similarity hypothesis

Kolmogorov's similarity hypothesis (1941)

- Andrey Nikolaevich Kolmogorov (1903–1987).
- Famous Russian mathematician.
- Very influential 1941 theory of homogeneous, isotropic, incompressible turbulence based on Richardson's ideas.



Kolmogorov's similarity hypothesis (1941)

Kolmogorov's theory of turbulence

- Turbulence displays universal properties independent of initial and boundary conditions.
- Energy is added to the fluid on the inertial scale ℓ_o and is dissipated as heat on the dissipative scale.
- Energy transfer between eddies on intermediate scales is lossless.

Kolmogorov's similarity hypothesis (1941)

Kolmogorov's local isotropy hypothesis

- As Re becomes large, the small-scale turbulent motions in all flows have similar universal character.
- These motions are assumed to be statistically isotropic (local isotropy hypothesis),

Kolmogorov's similarity hypothesis (1941)

Kolmogorov's first hypothesis

- Smallest scales receive energy at a rate proportional to the dissipation of energy rate.
- Motion of the very smallest scales in a flow depend only on:
  - rate of energy transfer from small scales

$$\epsilon \left[\frac{L^2}{T^3}\right]$$

  - kinematic viscosity

$$\nu \left[\frac{L^2}{T}\right]$$

Kolmogorov's similarity hypothesis (1941)

Using these, he defined the Kolmogorov scales (dissipation scales)

- length scale

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$$

- time scale

$$\tau = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}$$

- velocity scale

$$v = \frac{\eta}{\nu} = (\nu\epsilon)^{\frac{1}{4}}$$

Check units for yourself.

Kolmogorov's similarity hypothesis (1941)

- Recall, that the Reynolds number (Re$=UL/\nu$) is the ratio of inertia to viscous forces.
- Based on the Kolmogorov scales:

$$\text{Re} = \frac{\upsilon\eta}{\nu} = \frac{(\nu\epsilon)^{\frac{1}{4}}\left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}}{\nu} = \nu^{\frac{1}{4}}\epsilon^{\frac{1}{4}}\nu^{\frac{3}{4}}\epsilon^{-\frac{1}{4}}\nu^{-1} = 1$$

Or in other words, the Kolmogorov length scale is the scale at which Re=1

Kolmogorov's similarity hypothesis (1941)

- From these scales, we can also form the ratios of the largest to smallest scales in a flow.
- We will denote the largest length, time, and velocity scales as ℓ_o, t_o, and U_o, respectively.
- We can approximate dissipation at large scales as

$$\epsilon \sim \frac{U_o^3}{\ell_o}$$

Kolmogorov's similarity hypothesis (1941)

- length scale

$$
\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} \sim \left(\frac{\nu^3 \ell_o}{U_o^3}\right)^{\frac{1}{4}}
$$

$$
\Rightarrow \frac{\ell_o^{\frac{1}{4}}}{\eta} \sim \frac{U_o^{\frac{3}{4}}}{\nu^{\frac{3}{4}}}
$$

$$
\Rightarrow \frac{\ell_o}{\eta} \sim \frac{U_o^{\frac{3}{4}} \ell_o^{\frac{3}{4}}}{\nu^{\frac{3}{4}}}
$$

$$
\boxed{\Rightarrow \frac{\ell_o}{\eta} \sim \mathsf{Re}^{\frac{3}{4}}}
$$

Kolmogorov's similarity hypothesis (1941)

- velocity scale

$$v = \frac{\eta}{\nu} \sim \left(\frac{\nu U_o^3}{\ell_o}\right)^{\frac{1}{4}}$$

$$\Rightarrow \frac{U_0^{\frac{3}{4}}}{v} \sim \frac{\ell_o^{\frac{1}{4}}}{\nu^{\frac{1}{4}}}$$

$$\Rightarrow \frac{U_0}{v} \sim \frac{U_o^{\frac{1}{4}} \ell_o^{\frac{1}{4}}}{\nu^{\frac{1}{4}}}$$

$$\boxed{\Rightarrow \frac{U_0}{v} \sim \mathsf{Re}^{\frac{1}{4}}}$$

Kolmogorov's similarity hypothesis (1941)

- time scale

$$
\begin{aligned}
\tau &= \frac{\eta}{v} \\
&\Rightarrow \frac{t_o}{\tau} = \frac{\ell_o / U_o}{\eta / v} \\
&\Rightarrow \frac{t_o}{\tau} = \left(\frac{\ell_o}{\eta}\right)\left(\frac{U_o}{v}\right)^{-1} \\
&\Rightarrow \frac{t_o}{\tau} \sim \mathsf{Re}^{\frac{3}{4}} \mathsf{Re}^{-\frac{1}{4}} \\
&\boxed{\Rightarrow \frac{t_o}{\tau} \sim \mathsf{Re}^{\frac{1}{2}}}
\end{aligned}
$$

Kolmogorov's similarity hypothesis (1941)

- For very high-Re flows (*e.g.*, Atmosphere), we have a range of scales that is small compared to ℓ_o but large compared to η.
- As Re increases, ℓ_o/η increases. This results in a larger separation of between large and small scales.

Kolmogorov's similarity hypothesis (1941)

- Consider typical atmospheric scales:

$$U_o \sim 10\ \mathrm{m\ s^{-1}},\ \ell_o \sim 10^3\ \mathrm{m},\ \nu \sim 10^{-5}\ \mathrm{m^2\ s^{-1}}$$

- which gives us,

$$\mathsf{Re} = \frac{U_o \ell_o}{\nu} \sim \frac{(10\ \mathrm{m\ s^{-1}})(10^3\ \mathrm{m})}{10^{-5}\ \mathrm{m^2\ s^{-1}}} \sim 10^9$$

- thus,

$$\begin{aligned}
\eta &\sim \ell_o \mathsf{Re}^{-\frac{3}{4}} \sim 0.00018\ \mathrm{m} \\
\upsilon &\sim U_o \mathsf{Re}^{-\frac{1}{4}} \sim 0.06\ \mathrm{m\ s^{-1}} \\
\tau &\sim \frac{\ell_o}{U_o} \mathsf{Re}^{-\frac{1}{2}} \sim 0.003\ \mathrm{s}
\end{aligned}$$

You can start to see why explicitly resolving all scales in a typical atmosphere is expensive!

Kolmogorov's similarity hypothesis (1941)

Kolmogorov's second hypothesis

- In turbulent flow, a range of scales exists at very high Re where statistics of motion in a range l ($\ell_o \gg \ell \gg \eta$) have a universal form that is determined only by ϵ (dissipation) and independent of ν (kinematic viscosity).
- Kolmogorov formed his hypothesis and examined it by looking at the PDF of velocity increments Δu.

Kolmogorov's similarity hypothesis (1941)

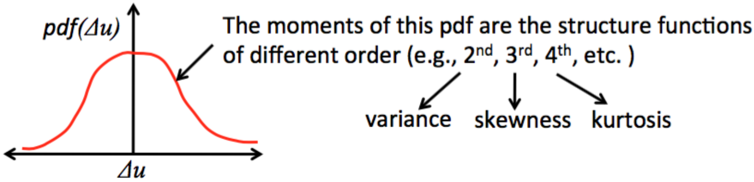
Kolmogorov's second hypothesis

- Using dimensional analysis (Buckingham Pi), we have that the spectral energy density, $[L^3/T^2]$, is a function of only wavenumber $k = 2\pi/\ell$, $[1/L]$, and dissipation ϵ, $[L^2/T^3]$:
- We arrive at:

$$E(K) = c_k \epsilon^{2/3} k^{-5/3}$$

  where c_k is the Kolmogorov "constant" and is assumed to be around 1.5 or so.
- This is the famous Kolmogorov's -5/3 power law.

Kolmogorov's similarity hypothesis (1941)



What are structure functions? The PDF? Let's quickly recap statistics and how they tie in to scales.

Stats review

- The PDF is the integral of the CDF
- It gives the probability per unit distance in the sample space – hence, the term *density*
- If two or more signals have the same PDF, then they are considered to be statistically identical.
- Practically speaking, we find the PDF of a time (or space) series by:
  - Create a histogram of the series(group values into bins)
  - Normalize the bin weights by the total # of points

Stats review

Autocovariance measures how a variable changes with different lags, s.

$$R(s) \equiv \langle u(t)u(t+s)\rangle$$

or the *autocorrelation function*

$$\rho(s) \equiv \frac{\langle u(t)u(t+s)\rangle}{u(t)^2}$$

Or for the discrete form

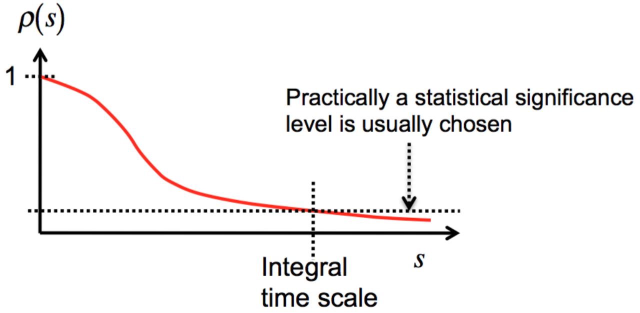
$$\rho(s_j) \equiv \frac{\sum_{k=0}^{N-j-1}(u_k u_{k+j})}{\sum_{k=0}^{N-1}(u_k^2)}$$

Notes on autocovariance and autocorrelation

- These are very similar to the covariance and correlation coefficient
- The difference is that we are now looking at the linear correlation of a signal with itself but at two different times (or spatial points), i.e. we lag the series.
- We could also look at the cross correlations in the same manner (between two different variables with a lag).
- $\rho(0) = 1$ and $|\rho(s)| \leq 1$

Stats review

- In turbulent flows, we expect the correlation to diminish with increasing time (or distance) between points
- We can use this to define an integral time (or space) scale. It is defined as the time lag where the integral $\int \rho(s) ds$ converges.
- It can also be used to define the largest scales of motion (statistically).



The *structure function* is another important two-point statistic.

$$D_n(r) \equiv \langle [U_1(x + r, t) - U_1(x, t)]^n \rangle$$

- This gives us the average difference between two points separated by a distance r raised to a power n.
- In some sense it is a measure of the moments of the velocity increment PDF.
- Note the difference between this and the autocorrelation which is statistical linear correlation (*i.e.*, multiplication) of the two points.

Fourier transforms

Alternatively, we can also look at turbulence in wave (frequency) space. **Fourier transforms** are a common tool in fluid dynamics (see Pope, Appendix D-G, Stull handouts online).

Some uses:

- Analysis of turbulent flow
- Numerical simulations of N-S equations
- Analysis of numerical schemes (modified wavenumbers)

Fourier transforms

- Consider a periodic function $f(x)$ [could also be $f(t)$] on a domain of length 2π.
- The Fourier representation of this function (or a general signal) is:

$$f(x) = \sum_{k=-\infty}^{k=\infty} \hat{f}_k e^{ikx}$$

where k is the wavenumber (frequency if $f(t)$), and $\hat{f}_k$ are the Fourier coefficients which in general are complex.

Fourier transforms

Why pick e^{ikx}?

- Orthogonality

$$\int_0^{2\pi} e^{i(k-k')x} dx = \begin{cases} 0, & \text{if } k \neq k' \\ 2\pi & \text{if } k = k' \end{cases}$$

- a big advantage of orthogonality is independence between Fourier modes
- e^{ix} is independent of e^{i2x}, just like we have with Cartesian coordinates – where i, j, k are all independent of each other

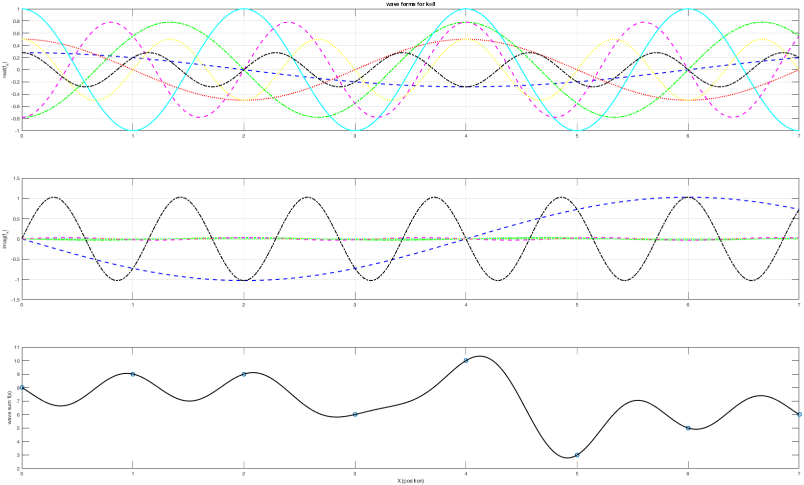
Fourier transforms

What are we doing?

- Recall from Euler's formula that $e^{ix} = \cos(x) - i\sin(x)$
- The Fourier transform decomposes a signal (space or time) into sine and cosine wave components of different amplitudes and wave numbers (or frequencies).

Fourier transforms

Fourier transform example (from Stull, see FourierTransDemo.m)



Fourier transforms

- The Fourier representation below is a representaion of a series as a function of sine and cosine waves. It takes $f(x)$ and transforms it into wave space.
- Fourier transform pair: consider a periodic function on a domain of 2π

$$f_k = F\{f(x)\} \equiv \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx \quad \rightarrow \text{forward transform}$$

$$f(x) = F\{\hat{f}_k\}^{-1} \equiv \sum_{k=-\infty}^{k=\infty} \hat{f}_k e^{ikx} \quad \rightarrow \text{backward transform}$$

- The forward transform moves us into Fourier (or wave) space and the backward transform moves us from wave space back to real space.

Fourier transforms

An alternative form of the Fourier transform (using Euler's) is:

$$f(x) = a_0 + \sum_{k=-\infty}^{k=\infty} a_k \cos(kx) - b_k \sin kx$$

where a_k and b_k are the real and imaginary components of f_k, respectively.

Fourier transform properties

- if $f(x)$ is real, then:

$$\hat{f}_k = \hat{f}_k A^*$$

- Parseval's Theorem:

$$\frac{1}{2\pi}\int_0^{2\pi} f(x) f^*(x) dx = \sum_{k=-\infty}^{k=\infty} \hat{f}_k \hat{f}_k^*$$

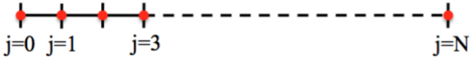
- The Fourier representation is the best possible representation for $f(x)$ in the sense that the error:

$$\mathsf{e} = \int_0^{2\pi} \left| f\left(x - \sum_{k=-N}^{N} c_k e^{ikx}\right)\right|^2 dx$$

is a minimum when $c_k = \hat{f}_k$

Discrete Fourier transform

- Consider the periodic function f_j on the domain $0 \leq x \leq L$ (periodicity implies that $f(0) = f(N)$)



with $x_j = jh$ and $h = L/N$

- Discrete Fourier representation:

$$f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{i\frac{2\pi}{L}kx_j} \Rightarrow \text{backward (inverse) transform}$$

  We know f_j at N pts, don't know $\hat{f}_k$ at k values (N of them).

- Using discrete orthogonality:

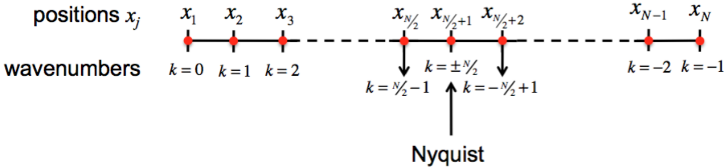
$$\hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-i\frac{2\pi}{L}kx_j} \Rightarrow \text{forward transform}$$

Discrete Fourier transform

- Discrete Fourier Transform (DFT) example and more explanation found on the website/Canvas (Stull, Chapter 8.4–8.6., Pope appendix F, FourierTransDemo.m).
- Implementation of DFT by brute force $\rightarrow \mathcal{O}(N^2)$ operations.
- In practice, we almost always use a Fast Fourier Transform (FFT) $\rightarrow \mathcal{O}(N \log_2 N)$ operations.

Discrete Fourier transform

- Almost all FFT routines (*e.g.*, Matlab, FFTW, Intel, Numerical Recipes, etc.) save their data with the following format:



Fourier transform applications: autocorrelation

Autocorrelation

- We can use the discrete Fourier Transform to speed up the autocorrelation calculation (or in general any cross-correlation with a lag). Discretely,

$$R_{ff}(s_l) = \sum_{j=0}^{N-1} f(x_j) f(x_j + s_l) \ \Rightarrow \mathcal{O}(N^2)\text{operations}$$

- If we express R_{ff} as a Fourier series

$$R_{ff}(s_l) = \sum\nolimits_k \hat{R}_{ff} e^{iks_l} \Rightarrow R_f f(0) = \sum_k \hat{R}_{ff}$$

and we can show that

$$R_{ff}(0) = \sum\nolimits_k N \underbrace{|\hat{f}_k|^2}_{\text{magnitude of Fourier Coefficients}}$$

Fourier transform applications: autocorrelation

How can we interpret this?

- In physical space

$$R_{ff}(0) = \sum_{j=0}^{N-1} f_j^2 \text{ (\textit{i.e.}, the mean variance)}$$

$$\Rightarrow \sum_{j=0}^{N-1} f_j^2 = \sum_{k=-N/2}^{N/2-1} \underbrace{N|\hat{f}_k|^2}_{\substack{\text{energy}\\ \text{spectral density}}} \Big\} \substack{\text{total contribution}\\ \text{to the variance}}$$

Fourier transform applications: spectrum

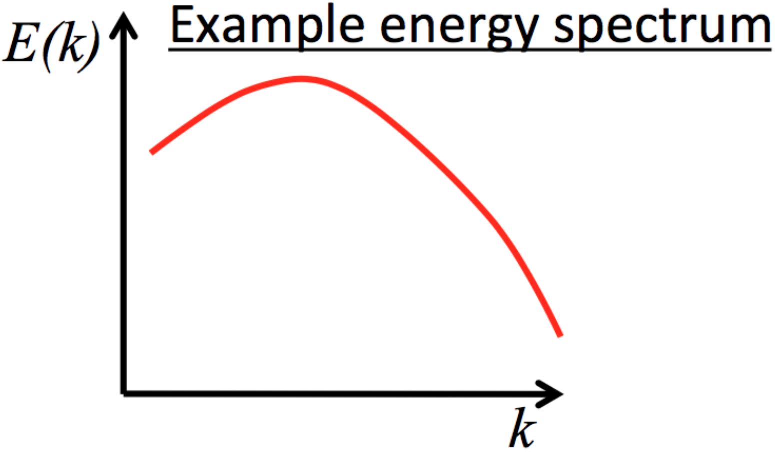
Energy Spectrum: (power spectrum, energy spectral density)

- If we look at specific k values from we can define:

$$E(k) = N|\hat{f}_k|^2$$

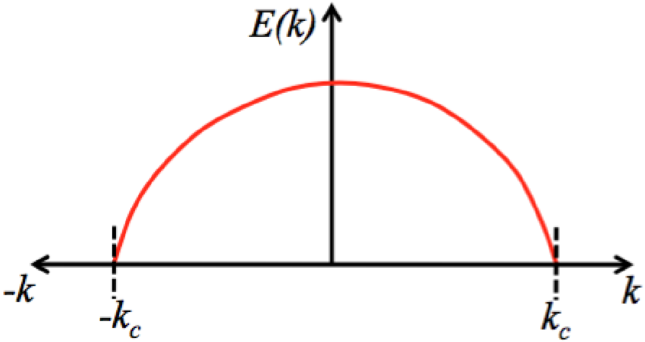
  where $E(k)$ is the energy spectral density
- The square of the Fourier coefficients is the contribution to the variance by fluctuations of scale k (wavenumber or equivalently frequency)
- Typically (when written as) $E(k)$ we mean the contribution to the turbulent kinetic energy (TKE) $= 0.5(u^2 + v^2 + w^2)$ and we would say that $E(k)$ is the contribution to TKE for motions of the scale (or size) k . For a single velocity component in one direction we would write $E_{11}(k_1)$.

Example energy spectrum



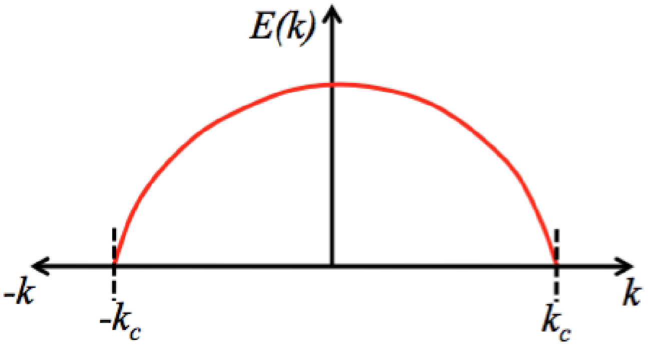
Spectrum: sampling theorem

- Band-Limited function: a function where $\hat{f}_k = 0$ for $|k| > k_c$.



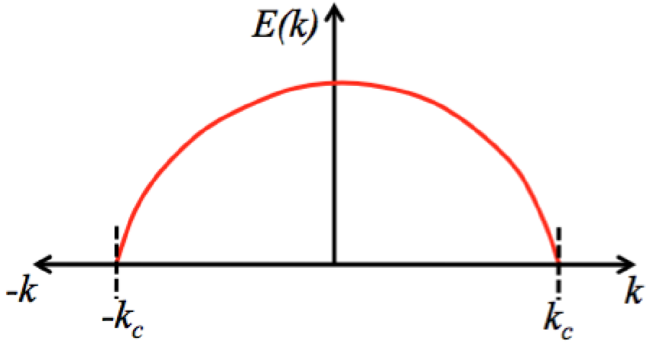
Spectrum: sampling theorem

- <u>Theorem</u>: if $f(x)$ is band-limited, then $f(x)$ is completely represented by its values on a discrete grid, $x_n = n\pi/k_c$, where n is an integer $(\infty < n < \infty)$ and k_c is called the Nyquist frequency.



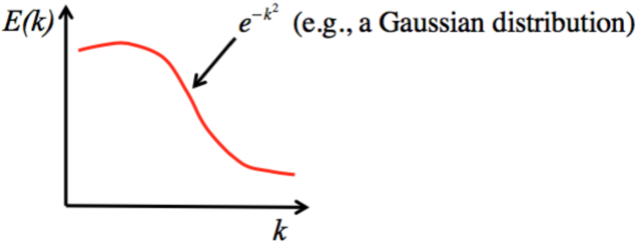
Spectrum: sampling theorem

- Implication: if we have $x_j = j\pi/k_c = jh$ $(h = \pi/k_c)$ with a domain of 2π, then $h = 2\pi/N = \pi/k_c \Rightarrow k_c = N/2$
- If the number of points is $\geq 2k_c$, then the *discrete Fourier transform is the exact solution*. For example, if $f(x) = \cos(6x)$, then we need $N \geq 12$ points to represent the function exactly.



Spectrum: sampling theorem

- What if $f(x)$ is not band-limited?
- What if $f(x)$ is band-limited, but sampled at a rate $< k_c$ (e.g., $f(x) = cos(6x)$ with 8 points)?



- The result is aliasing $\rightarrow$ contamination of resolved energy by energy outside of the resolved scales.

Spectrum: aliasing

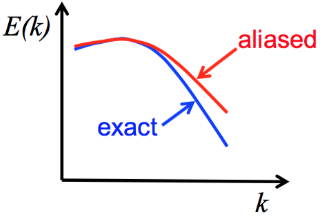
- Consider $e^{ik_1 x_j}$ and $e^{ik_2 x_j}$ and let $k_1 = k_2 + 2mk_c$, where k_c is the Nyquist frequency, $m = \pm$ any integer, and $x_j = j\pi/k_c$:

$$
\begin{aligned}
e^{ik_1 x_j} &= e^{i(k_2+2mk_c)x_j} \\
&= e^{ik_2 x_j} e^{2mk_c x_j} \\
&= e^{ik_2 x_j} e^{2mk_c j\pi/k_c} \\
&= e^{ik_2 x_j} \underbrace{e^{i2\pi mj}}_{=1,\text{ integer fn of } 2\pi} \\
e^{ik_1 x_j} &= e^{ik_2 x_j}
\end{aligned}
$$

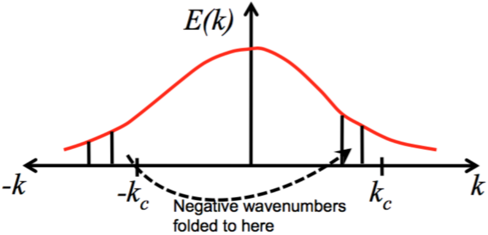
The result is that we cannot distinguish between k_2 and $k_1 = k_2 + 2mk_c$ on a discrete grid. k_1 is *aliased* onto k_2.

Spectrum: aliasing

- What does this mean for spectra?



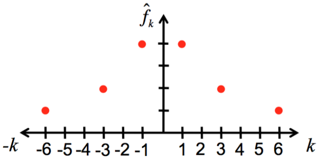
- What is actually happening?



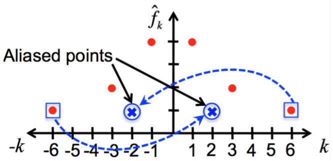
Spectrum: aliasing

Consider a function: $f(x) = \cos(x) + 0.5\cos(3x) + 0.25\cos(6x)$

- Fourier coefficients (all real)



- Consider $N = 8 \rightarrow k_c = 4$



- Aliasing, if $m = 1$, $\Rightarrow k_1 = k_2 + 2mk_c = k_2 + 8m \Rightarrow -6$ gets aliased to 2. If $m = -1, k_1 = k_2 - 8 \Rightarrow 6$ gets aliased to -2.

Spectrum: aliasing

- Aliasing decreases if N (sampling rate) increases.
- For more on Fourier Transforms see Pope Ch. 6, online handout from Stull, or Press et al., Ch 12-13.

Spectrum and Kolmogorov

Back to Kolmogorov

- Another way to look at this (equivalent to structure functions) is to examine what it means for $E(k)$ where $E(k)dk =$TKE contained between k and $k + dk$.
- What are the implications of Kolmogorov's hypothesis for $E(k)$? – K41$\Rightarrow$ $E(k) = f(k, \epsilon)$
- By dimensional analysis we can find that:

$$E(K) = c_k \epsilon^{2/3} k^{-5/3}$$

  Kolmogorov's 5/3 power law.
- This expression is valid for the range of length scales ℓ where $\ell_o \gg \ell \gg \eta$ and is usually called the inertial subrange of turbulence.

Spectrum and Kolmogorov

Example energy spectrum