Environmental Fluid Dynamics: Lecture 17

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Spring 2017

Overview

1. Turbulent Momentum Flux Balance
2. Turbulent Buoyancy Flux Balance

Turbulent Momentum Flux Balance

Turbulent Momentum Flux Balance

- We will derive the turbulent momentum flux balance equation.
- We write the equation in terms of buoyancy b (Lecture 11):

$$b = \beta\theta'_v$$

where $\beta = g/\theta_0$ is the buoyancy parameter, g is gravity, and θ_0 is a constant reference potential temperature (say $300\ \mathrm{K}$).

- We also write the equation in terms of normalized pressure Π

$$\Pi = \frac{p - p_0}{\rho_0} = \frac{p'}{\rho_0}$$

where p_0 is the reference (or base-state) pressure and ρ_0 is the reference density.

Turbulent Momentum Flux Balance

- We start with the momentum balance equation written in indicial notation and neglect rotational effects

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_j u_i)}{\partial x_j} = -\frac{\partial \Pi}{\partial x_i} + b\delta_{i3} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \tag{1}$$

- Now we decompose into mean and perturbation parts

$$\frac{\partial (\overline{u_i} + u_i')}{\partial t} + \frac{\partial (\overline{u_j} + u_j')(\overline{u_i} + u_i')}{\partial x_j} = -\frac{\partial (\overline{\Pi} + \Pi')}{\partial x_i} + (\bar{b} + b')\delta_{i3} + \nu \frac{\partial^2 (\overline{u_i} + u_i')}{\partial x_j^2}$$

Turbulent Momentum Flux Balance

- Next, we can simplify and then apply Reynolds averaging

$$\frac{\partial \overline{\overline{u_i}}}{\partial t} + \frac{\partial \overline{u_i'}}{\partial t} + \frac{\partial \overline{\overline{u_j}\ \overline{u_i}}}{\partial x_j} + \frac{\partial \overline{\overline{u_j} u_i'}}{\partial x_j} + \frac{\partial \overline{u_j' \overline{u_i}}}{\partial x_j} + \frac{\partial \overline{u_j' u_i'}}{\partial x_j}$$
$$= -\frac{\partial \overline{\overline{\Pi}}}{\partial x_i} - \frac{\partial \overline{\Pi'}}{\partial x_i} + \overline{\overline{b}}\delta_{i3} + \overline{b'}\delta_{i3} + \nu \frac{\partial^2 \overline{\overline{u_i}}}{\partial x_j^2} + \nu \frac{\partial^2 \overline{u_i'}}{\partial x_j^2}$$

- Applying the rules of averaging yields

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial \overline{u_j}\ \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j' u_i'}}{\partial x_j} = -\frac{\partial \overline{\Pi}}{\partial x_i} + \overline{b}\delta_{i3} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2} \qquad (2)$$

Turbulent Momentum Flux Balance

- Subtract Eq. (2) from Eq. (1) to obtain an expression for u_i'

$$\frac{\partial(u_i - \overline{u_i})}{\partial t} + \frac{\partial(u_j u_i - \overline{u_j}\;\overline{u_i} - \overline{u_j' u_i'})}{\partial x_j} = -\frac{\partial(\Pi - \overline{\Pi})}{\partial x_i} + (b - \overline{b})\delta_{i3} + \nu \frac{\partial^2 (u_i - \overline{u_i})}{\partial x_j^2}$$

- Expanding terms and simplifying yields

$$\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_j}\left[(\overline{u_j} + u_j')(\overline{u_i} + u_i') - \overline{u_j}\;\overline{u_i} - \overline{u_j' u_i'}\right] = -\frac{\partial \Pi'}{\partial x_i} + b'\delta_{i3} + \nu \frac{\partial^2 u_i'}{\partial x_j^2}$$

Turbulent Momentum Flux Balance

- Final rearrangement leads to the following expression for u_i'

$$\boxed{\begin{aligned}\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_j}\left[u_j'\overline{u_i} + \overline{u_j}u_i' + u_j'u_i' - \overline{u_j'u_i'}\right] = -\frac{\partial \Pi'}{\partial x_i} + b'\delta_{i3} \\ + \nu\frac{\partial^2 u_i'}{\partial x_j^2}\end{aligned}} \tag{3}$$

- Now we want to derive an expression in flux form

Turbulent Momentum Flux Balance

- First, multiply Eq. (3) by u'_k

$$u'_k \frac{\partial u'_i}{\partial t} + u'_k \frac{\partial u'_j \overline{u_i}}{\partial x_j} + u'_k \frac{\partial \overline{u_j} u'_i}{\partial x_j} + u'_k \frac{\partial u'_j u'_i}{\partial x_j} - u'_k \frac{\partial \overline{u'_j u'_i}}{\partial x_j} \qquad (4)$$

$$= -u'_k \frac{\partial \Pi'}{\partial x_i} + u'_k b' \delta_{i3} + \nu u'_k \frac{\partial^2 u'_i}{\partial x_j^2}$$

- Now, interchange i and k

$$u'_i \frac{\partial u'_k}{\partial t} + u'_i \frac{\partial u'_j \overline{u_k}}{\partial x_j} + u'_i \frac{\partial \overline{u_j} u'_k}{\partial x_j} + u'_i \frac{\partial u'_j u'_k}{\partial x_j} - u'_i \frac{\partial \overline{u'_j u'_k}}{\partial x_j} \qquad (5)$$

$$= -u'_i \frac{\partial \Pi'}{\partial x_k} + u'_i b' \delta_{k3} + \nu u'_i \frac{\partial^2 u'_k}{\partial x_j^2}$$

Turbulent Momentum Flux Balance

- Next, add Eqs. (4) and (5)

$$
u'_k \frac{\partial u'_i}{\partial t} + u'_i \frac{\partial u'_k}{\partial t} + u'_k \frac{\partial u'_j \overline{u_i}}{\partial x_j} + u'_i \frac{\partial u'_j \overline{u_k}}{\partial x_j} + u'_k \frac{\partial \overline{u_j} u'_i}{\partial x_j} + u'_i \frac{\partial \overline{u_j} u'_k}{\partial x_j}
$$
$$
+ u'_k \frac{\partial u'_j u'_i}{\partial x_j} + u'_i \frac{\partial u'_j u'_k}{\partial x_j} - u'_k \frac{\partial \overline{u'_j u'_i}}{\partial x_j} - u'_i \frac{\partial \overline{u'_j u'_k}}{\partial x_j}
$$
$$
= -u'_k \frac{\partial \Pi'}{\partial x_i} - u'_i \frac{\partial \Pi'}{\partial x_k} + u'_k b' \delta_{i3} + u'_i b' \delta_{k3} + \nu u'_k \frac{\partial^2 u'_i}{\partial x_j^2} + \nu u'_i \frac{\partial^2 u'_k}{\partial x_j^2}
$$

- Use product rule and incompressibility condition $(\partial u_j / \partial x_j = 0)$

$$
\frac{\partial u'_k u'_i}{\partial t} + \overline{u_j} \frac{\partial u'_k u'_i}{\partial x_j} + u'_k u'_j \frac{\partial \overline{u_i}}{\partial x_j} + u'_j u'_i \frac{\partial \overline{u_k}}{\partial x_j} + \frac{\partial u'_k u'_j u'_i}{\partial x_j}
$$
$$
- u'_k \frac{\partial \overline{u'_j u'_i}}{\partial x_j} - u'_i \frac{\partial \overline{u'_k u'_j}}{\partial x_j} =
$$
$$
- u'_k \frac{\partial \Pi'}{\partial x_i} - u'_i \frac{\partial \Pi'}{\partial x_k} + u'_k b' \delta_{i3} + u'_i b' \delta_{k3} + \nu u'_k \frac{\partial^2 u'_i}{\partial x_j^2} + \nu u'_i \frac{\partial^2 u'_k}{\partial x_j^2}
$$

Turbulent Momentum Flux Balance

- Finally, apply Reynolds averaging and use the product rule to expand and rewrite the pressure and viscous terms.

$$
\begin{aligned}
\underbrace{\frac{\partial(\overline{u_k'u_i'})}{\partial t}}_{1} = & -\underbrace{\overline{u}_j\frac{\partial(\overline{u_k'u_i'})}{\partial x_j}}_{2} - \underbrace{\left[\overline{u_j'u_i'}\frac{\partial \overline{u}_k}{\partial x_j} + \overline{u_k'u_j'}\frac{\partial \overline{u}_i}{\partial x_j}\right]}_{3} - \underbrace{\frac{\partial(\overline{u_k'u_j'u_i'})}{\partial x_j}}_{4} \\
& + \underbrace{\overline{u_k'b'}\delta_{i3} + \overline{u_i'b'}\delta_{k3}}_{5} \\
& - \left[\underbrace{\frac{\partial(\overline{u_k'\Pi'})}{\partial x_i} + \frac{\partial(\overline{u_i'\Pi'})}{\partial x_k}}_{6} - \underbrace{\overline{\Pi'\left(\frac{\partial u_k'}{\partial x_i} + \frac{\partial u_i'}{\partial x_k}\right)}}_{7}\right] \\
& + \underbrace{\nu\frac{\partial^2(\overline{u_k'u_i'})}{\partial x_j^2}}_{8} - \underbrace{2\nu\overline{\frac{\partial u_k'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}}_{9}
\end{aligned}
\tag{6}
$$

Turbulent Momentum Flux Balance

Terms in Eq. (6)

1. Storage of momentum flux
2. Advection of momentum flux by the mean wind
3. Production of momentum flux by the mean wind shear
4. Transport of momentum flux by turbulence (turbulent diffusion)
5. Production/destruction of momentum flux by buoyancy
6. Transport of momentum flux by pressure (pressure diffusion)
7. Redistribution of momentum flux by pressure
8. Molecular diffusion of momentum flux
9. Viscous dissipation of momentum flux

Turbulent Buoyancy Flux Balance

Turbulent Buoyancy Flux Balance

- We will derive the turbulent buoyancy flux balance equation.
- We start with the momentum balance equation written in indicial notation and neglect rotational effects

$$\frac{\partial b}{\partial t} + \frac{\partial u_j b}{\partial x_j} = -N^2 u_j \delta_{j3} + \nu_h \frac{\partial^2 b}{\partial x_j^2} \tag{7}$$

- Now we decompose into mean and perturbation parts

$$\begin{aligned}\frac{\partial(\overline{b} + b')}{\partial t} + \frac{\partial(\overline{u_j} + u_j')(\overline{b} + b')}{\partial x_j} =& - N^2(\overline{u_j} + u_j')\delta_{j3} \\ &+ \nu_h \frac{\partial^2(\overline{b} + b')}{\partial x_j^2}\end{aligned}$$

Turbulent Buoyancy Flux Balance

- Next, expand and then apply Reynolds averaging

$$\frac{\partial\overline{\overline{b}}}{\partial t}+\frac{\partial\overline{b'}}{\partial t}+\frac{\partial\overline{\overline{u_j}\ \overline{b}}}{\partial x_j}+\frac{\partial\overline{\overline{u_j}b'}}{\partial x_j}+\frac{\partial\overline{u_j'\overline{b}}}{\partial x_j}+\frac{\partial\overline{u_j'b'}}{\partial x_j}$$
$$=-N^2\overline{\overline{u_j}}\delta_{j3}-N^2\overline{u_j'}\delta_{j3}+\nu_h\frac{\partial^2\overline{\overline{b}}}{\partial x_j^2}+\frac{\partial^2\overline{b'}}{\partial x_j^2}$$

- Applying the rules of averaging yields

$$\frac{\partial\overline{b}}{\partial t}+\frac{\partial\overline{u_j}\ \overline{b}}{\partial x_j}+\frac{\partial\overline{u_j'b'}}{\partial x_j}=-N^2\overline{u_j}\delta_{j3}+\nu_h\frac{\partial^2\overline{b}}{\partial x_j^2} \tag{8}$$

Turbulent Buoyancy Flux Balance

- Subtract Eq. (8) from Eq. (7) to obtain an expression for b'

$$\frac{\partial(b-\bar{b})}{\partial t}+\frac{\partial}{\partial x_j}\left[u_j b-\overline{u_j}\bar{b}-\overline{u_j' b'}\right]=-N^2(u_j-\overline{u_j})\delta_{j3}+\nu_h\frac{\partial^2 b'}{\partial x_j^2}$$

$$\frac{\partial b'}{\partial t}+\frac{\partial}{\partial x_j}\left[(\overline{u_j}-u_j')(\bar{b}+b')-\overline{u_j}\bar{b}-\overline{u_j' b'}\right]=-N^2u_j'\delta_{j3}+\nu_h\frac{\partial^2 b'}{\partial x_j^2}$$

$$\boxed{\frac{\partial b'}{\partial t}+\frac{\partial}{\partial x_j}\left[\overline{u_j}b'+u_j'\bar{b}+u_j'b'-\overline{u_j' b'}\right]=-N^2u_j'\delta_{j3}+\nu_h\frac{\partial^2 b'}{\partial x_j^2}} \tag{9}$$

- Now we want to derive an expression in flux form

Turbulent Buoyancy Flux Balance

- First, multiply Eq. (3) by b'

$$b'\frac{\partial u_i'}{\partial t} + b'\frac{\partial}{\partial x_j}\left[u_j'\overline{u_i} + \overline{u_j}u_i' + u_j'u_i' - \overline{u_j'u_i'}\right] = -b'\frac{\partial \Pi'}{\partial x_i} + b'b'\delta_{i3} + \nu b'\frac{\partial^2 u_i'}{\partial x_j^2} \quad (10)$$

- Next, multiply Eq. (9) by u_i'

$$u_i'\frac{\partial b'}{\partial t} + u_i'\frac{\partial}{\partial x_j}\left[\overline{u_j}b' + u_j'\overline{b} + u_j'b' - \overline{u_j'b'}\right] = -N^2 u_j'u_i'\delta_{j3} + \nu_h u_i'\frac{\partial^2 b'}{\partial x_j^2} \quad (11)$$

Turbulent Buoyancy Flux Balance

- Add Eqs. (10) by (11)

$$
\begin{aligned}
b' \frac{\partial u_i'}{\partial t} + u_i' \frac{\partial b'}{\partial t} + b' u_j' \frac{\partial \overline{u_i}}{\partial x_j} + u_i' \overline{u_j} \frac{\partial b'}{\partial x_j} + b' \overline{u_j} \frac{\partial u_i'}{\partial x_j} \\
+ u_i' u_j' \frac{\partial \bar{b}}{\partial x_j} + b' u_j' \frac{\partial u_i'}{\partial x_j} + u_i' u_j' \frac{\partial b'}{\partial x_j} - b' \frac{\partial \overline{u_j' u_i'}}{\partial x_j} - u_i' \frac{\partial \overline{u_j' b'}}{\partial x_j} \\
= -b' \frac{\partial \Pi'}{\partial x_i} + b' b' \delta_{i3} - N^2 u_j' u_i' \delta_{j3} + \nu b' \frac{\partial^2 u_i'}{\partial x_j^2} + \nu_h u_i' \frac{\partial^2 b'}{\partial x_j^2}
\end{aligned}
$$

- Use product rule and incompressibility condition $(\partial u_j / \partial x_j = 0)$

$$
\begin{aligned}
\frac{\partial u_i' b'}{\partial t} + \overline{u_j} \frac{\partial u_i' b'}{\partial x_j} = - \left(u_j' b' \frac{\partial \overline{u_i}}{\partial x_j} + u_j' u_i' \frac{\partial \bar{b}}{\partial x_j} \right) - \frac{\partial u_j' u_i' b'}{\partial x_j} \\
+ b' \frac{\partial \overline{u_j' u_i'}}{\partial x_j} + u_i' \frac{\partial \overline{u_j' b'}}{\partial x_j} - b' \frac{\partial \Pi'}{\partial x_i} \\
+ b' b' \delta_{i3} - N^2 u_j' u_i' \delta_{j3} + \nu b' \frac{\partial^2 u_i'}{\partial x_j^2} + \nu_h u_i' \frac{\partial^2 b'}{\partial x_j^2}
\end{aligned}
$$

Turbulent Buoyancy Flux Balance

- Finally, apply Reynolds averaging and use the product rule to expand and rewrite the pressure and viscous terms.

$$\boxed{\begin{aligned} \underbrace{\frac{\partial \overline{u_i' b'}}{\partial t}}_{1} = & -\underbrace{\overline{u}_j \frac{\partial \overline{u_i' b'}}{\partial x_j}}_{2} - \underbrace{\overline{u_j' b'} \frac{\partial \overline{u_i}}{\partial x_j} - \overline{u_j' u_i'} \left[\frac{\partial \overline{b}}{\partial x_j} + N^2 \delta_{j3} \right]}_{3} \\ & - \underbrace{\frac{\partial \overline{u_j' u_i' b'}}{\partial x_j}}_{4} + \underbrace{\overline{b' b'} \delta_{i3}}_{5} - \underbrace{\left[\frac{\partial \overline{b' \Pi'}}{\partial x_i}\right.}_{6} - \underbrace{\left.\overline{\Pi' \frac{\partial b'}{\partial x_i}} \right]}_{7} \\ & + \underbrace{\nu \frac{\partial^2 \overline{u_i' b'}}{\partial x_j^2}}_{8} - \underbrace{2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial b'}{\partial x_j}}}_{9} \end{aligned}} \quad (12)$$

Turbulent Buoyancy Flux Balance

Terms in Eq. (12)

1. Storage of buoyancy flux
2. Advection of buoyancy flux by the mean wind
3. Production of buoyancy flux by the mean wind and buoyancy shear + stratification
4. Transport of buoyancy flux by turbulence (turbulent diffusion)
5. Production/destruction of buoyancy flux by buoyancy
6. Transport of buoyancy flux by pressure (pressure diffusion)
7. Redistribution of buoyancy flux by pressure
8. Molecular diffusion of buoyancy flux
9. Viscous dissipation of buoyancy flux