Environmental Fluid Dynamics: Lecture 14

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Overview

1. Atmospheric Dynamics: Basic Equations
   - Summation Review
   - Mechanical Energy Equation
   - Thermal Energy Equation
2. Summary

Summation Review

Einstein Summation Review

Consider:

$$U_m \Rightarrow \text{generic velocity vector}$$
$$x_m \Rightarrow \text{generic component of distance}$$
$$e_m \Rightarrow \text{generic unit vector}$$

where $m = 1, 2, 3$. We can write the individual components as:

$$U_1 = u \qquad x_1 = x \qquad e_1 = \hat{i}$$
$$U_2 = v \qquad x_2 = x \qquad e_2 = \hat{j}$$
$$U_3 = w \qquad x_3 = x \qquad e_3 = \hat{k}$$

A variable with:

- no free indices is a **scalar**.
- one free indice is a **vector**.
- two free indices is a **tensor**.

Einstein Summation Review

There are two primary summation operators that we will use:

- **Kronecker Delta**

$$\delta_{mn} = \begin{cases} +1, & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

- **Levi-Civita (Alternating Unit) Tensor**

$$\epsilon_{mnq} = \begin{cases} +1, & \text{if } mnq = 123, 231, 312 & \text{even permutation} \\ -1 & \text{if } mnq = 321, 213, 132 & \text{odd permutation} \\ 0 & \text{if } m = n, n = q, q = m & \text{any two indices repeated} \end{cases}$$

Atmospheric Dynamics:

Mechanical Energy Equation

Mechanical Energy Equation

- The mechanical energy equation describes the rate of change of a fluid element's kinetic energy.
- To start, we take the dot product of the velocity vector with our momentum balance equation.

$$\vec{U} \cdot \left(\underbrace{\rho \frac{D\vec{U}}{Dt}}_{1} = \underbrace{-\vec{\nabla} p}_{2} - \underbrace{2\rho \vec{\Omega} \times \vec{U}}_{3} + \underbrace{\rho \vec{g}}_{4} + \underbrace{\mu \vec{\nabla}^2 \vec{U}}_{5} \right)$$

- Note: We've multiplied the momentum balance equation by ρ
- We will examine each term separately and then put together for our final expression.

Mechanical Energy Equation

Term 1

$$
\begin{aligned}
\vec{U} \cdot \rho \frac{D\vec{U}}{Dt} &= u_i \left(\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} \right) \\
&= \rho u_i \frac{\partial u_i}{\partial t} + \rho u_j u_i \frac{\partial u_i}{\partial x_j} \quad \text{use product rule} \\
&= \rho \frac{\partial \left(\frac{u_i u_i}{2} \right)}{\partial t} + \rho u_j \frac{\partial \left(\frac{u_i u_i}{2} \right)}{\partial x_j} \\
&\boxed{= \rho \frac{\partial E}{\partial t} + \rho u_j \frac{\partial E}{\partial x_j}}
\end{aligned}
$$

where $E = u_i^2/2$ is kinetic energy.

Term 2

$$\vec{U} \cdot \left(-\vec{\nabla} p \right) = -u_i \frac{\partial p}{\partial x_i} \quad \text{use product rule}$$

$$= -\left(\frac{\partial (p u_i)}{\partial x_i} + p \cancel{\frac{\partial u_i}{\partial x_i}} \right)$$

$$\boxed{= -\frac{\partial (p u_i)}{\partial x_i}}$$

Mechanical Energy Equation

Term 3

$$\vec{U} \cdot \left(-2\rho\vec{\Omega} \times \vec{U}\right)$$

Recall that the Coriolis has the following components:

- x: fv
- y: -fu
- z: 0

This corresponds to $\epsilon_{ij3} f u_j$. So,

$$\vec{U} \cdot \left(-2\rho\vec{\Omega} \times \vec{U}\right) = \boxed{\epsilon_{ij3} f u_i u_j}$$

However, using the rules of the alternating unit tensor, you can show this is equal to zero. Alternatively, you know $-2\rho\Omega \times \vec{U}$ is $\perp$ to $\vec{U}$. Thus, their dot product is zero. Why?

Coriolis force does no work!

Term 4

$$\vec{U} \cdot \vec{g} = u_i g_i$$

But if we assume that gravity only acts in the $\hat{k}$ direction, then we can make use of the Kronecker delta:

$$\vec{U} \cdot \rho \vec{g} = \boxed{\rho u_i g_i \delta_{i3}}$$

Term 5

$$
\begin{aligned}
\vec{U} \cdot \nu \vec{\nabla}^2 \vec{U} &= \mu u_i \frac{\partial^2 u_i}{\partial x_j^2} \\
&= \mu u_i \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} \\
&\boxed{= u_i \frac{\partial \tau_{ij}}{\partial x_j}}
\end{aligned}
$$

Mechanical Energy Equation

All Together

$$\underbrace{\rho\frac{\partial E}{\partial t}}_{1} + \underbrace{\rho u_j \frac{\partial E}{\partial x_j}}_{2} = \underbrace{-\frac{\partial (pu_i)}{\partial x_i}}_{3} + \underbrace{\rho u_i g_i \delta_{i3}}_{4} + \underbrace{u_i \frac{\partial \tau_{ij}}{\partial x_j}}_{5}$$

Terms

1. Storage of kinetic energy
2. Advection of kinetic energy by the bulk flow
3. Work done against gravity
4. Work done against viscous forces

Mechanical Energy Equation

- Note: we can expand the viscous dissipation term as:

$$\begin{aligned} u_i \frac{\partial \tau_{ij}}{\partial x_j} &= \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j} \\ &= \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} S_{ij} \end{aligned}$$

  where S_{ij} is the symmetric part of the velocity gradient.
- The first term is the total rate of work, or molecular diffusion.
- The second term is the deformation work, or viscous dissipation.

Atmospheric Dynamics:

Thermal Energy Equation

Thermal Energy Equation

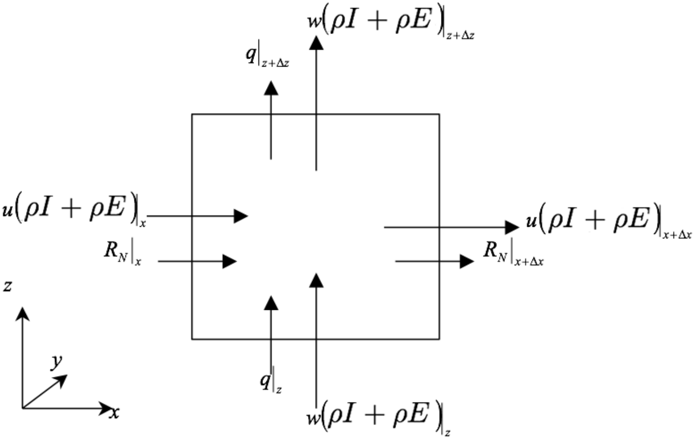
- We can write the 1st Law of Thermodynamics for an open unsteady system in words as follows, where total energy = internal (I) + kinetic ($E = V^2/2$):

$$\underbrace{\left\{\begin{array}{c}\text{rate of}\\ \text{total energy}\\ \text{accumulation}\end{array}\right\}}_{1} = \underbrace{\left\{\begin{array}{c}\text{net rate of}\\ \text{total energy in}\\ \text{by advection}\end{array}\right\}}_{2} + \underbrace{\left\{\begin{array}{c}\text{net rate of}\\ \text{heat added by}\\ \text{conduction}\end{array}\right\}}_{3}$$

$$+ \underbrace{\left\{\begin{array}{c}\text{net rate of}\\ \text{heat added by}\\ \text{radiation}\end{array}\right\}}_{4} + \underbrace{\left\{\begin{array}{c}\text{net rate of}\\ \text{heat added by}\\ \text{phase change}\end{array}\right\}}_{5}$$

$$- \underbrace{\left\{\begin{array}{c}\text{net rate of}\\ \text{work done by}\\ \text{fluid on}\\ \text{surroundings}\end{array}\right\}}_{6}$$

- This is seen visually as:



Let's look separately at each term in the previous description.

Thermal Energy Equation

Term 1

- Rate of accumulation of internal energy per unit volume and kinetic energy within the element (these are in units of W):

$$\Delta x \Delta y \Delta z \frac{\partial}{\partial t}(\rho I + \rho E)$$

Thermal Energy Equation

Term 2

- Net rate of advection of internal and kinetic Energy into the volume element:

$$
\begin{aligned}
&\Delta y \Delta z \left\{ u(\rho I + \rho E)_{x+\Delta x} - u(\rho I + \rho E)_{x} \right\} \\
+&\Delta x \Delta z \left\{ v(\rho I + \rho E)_{y+\Delta y} - v(\rho I + \rho E)_{y} \right\} \\
+&\Delta x \Delta y \left\{ w(\rho I + \rho E)_{z+\Delta z} - w(\rho I + \rho E)_{z} \right\}
\end{aligned}
$$

Thermal Energy Equation

Term 3

- Net rate of energy input by conduction into the volume element (molecular):

$$\Delta y \Delta z (q_x|_x - q_x|_{x+\Delta x}) + \Delta x \Delta z (q_y|_y - q_y|_{y+\Delta y}) + \Delta x \Delta y (q_z|_z - q_z|_{z+\Delta z})$$

Thermal Energy Equation

Term 4

- Net rate of energy input by all wavelengths of radiation into the volume element:

$$\Delta y \Delta z (R_{nx}|_x - R_{nx}|_{x+\Delta x}) + \Delta x \Delta z (R_{ny}|_y - R_{ny}|_{y+\Delta y}) \\ + \Delta x \Delta y (R_{nz}|_z - R_{nz}|_{z+\Delta z})$$

Thermal Energy Equation

Term 5

- Net rate of energy input by phase change into the volume element:

$$\Delta x \Delta y \Delta z (L_v \epsilon)$$

  where $L_v (\sim 2.45 \times 10^6 \text{ J kg}^{-1})$ is the latent heat of vaporization/condensation and $\epsilon (\text{kg m}^{-3} \text{ s}^{-1})$ is the evaporation/condensation rate per unit volume.
- Note: this is a body sink/source term.

Thermal Energy Equation

Term 6

- Net rate of work done by the fluid element against the surroundings.
- Recall that work rate done by a force is the magnitude of the force multiplied by the velocity in the direction of the force:

$$\dot{W} = \vec{F} \cdot \vec{U}$$

  So we will write the work rates as forces multiplied by velocities acting on our fluid element.
- The net rate of work will broken into work against body forces and work against surface forces.

Thermal Energy Equation

Term 6

- Work against body forces: rate of doing work against the gravitational force

$$\Delta x \Delta y \Delta z (\rho u g_x + \rho v g_y + \rho w g_z)$$

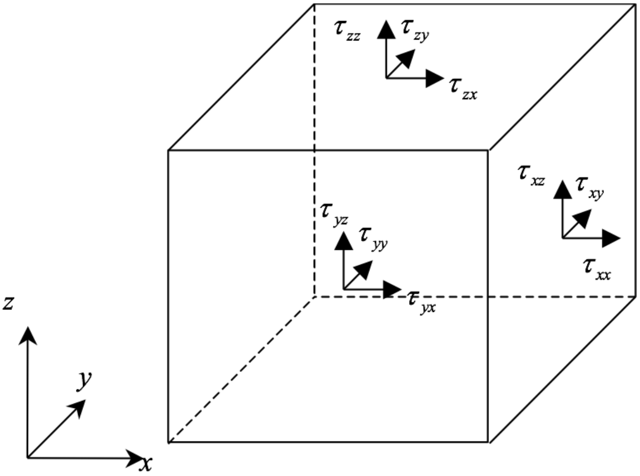
- Work against surface forces: rate of doing work against the pressure at the six faces of a volume element

$$\Delta y \Delta z \{(pu)_x|_{x+\Delta x} - (pu)_x|_x\} + \Delta x \Delta z \{(pv)_y|_{y+\Delta y} - (pv)_y|_y\} + \Delta x \Delta y \{(pw)_z|_{z+\Delta z} - (pw)_z|_z\}$$

Note that here we take the work rate to be $(p \cdot \vec{n} dA) \cdot \vec{U}$ where $\vec{n}$ is the outwardly pointing normal vector from the fluid element.

Term 6

- Work against surface forces: rate of doing work against the viscous force



Term 6

- Work against surface forces: rate of doing work against the viscous force

$$\begin{aligned}
&\Delta y \Delta z\{(\tau_{xx}u + \tau_{xy}v + \tau_{xz}w)_x - (\tau_{xx}u + \tau_{xy}v + \tau_{xz}w)_{x+\Delta x}\} \\
&+\Delta x \Delta z\{(\tau_{yx}u + \tau_{yy}v + \tau_{yz}w)_y - (\tau_{yx}u + \tau_{yy}v + \tau_{yz}w)_{y+\Delta y}\} \\
&+\Delta x \Delta y\{(\tau_{zx}u + \tau_{zy}v + \tau_{zz}w)_z - (\tau_{zx}u + \tau_{zy}v + \tau_{zz}w)_{z+\Delta z}\}
\end{aligned}$$

All Together

- Now we combine all terms, divide by $\Delta x \delta y \Delta z$, and take the limit as $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$ to obtain the **complete energy equation**:

$$
\begin{aligned}
\frac{\partial(\rho I + \rho E)}{\partial t} = & -\left(u\frac{\partial(\rho I + \rho E)}{\partial x} + v\frac{\partial(\rho I + \rho E)}{\partial y} + w\frac{\partial(\rho I + \rho E)}{\partial z}\right) \\
& - \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_x}{\partial z}\right) - \left(\frac{\partial pu}{\partial x} + \frac{\partial pv}{\partial y} + \frac{\partial pw}{\partial z}\right) \\
& - \left(\frac{\partial R_{nx}}{\partial x} + \frac{\partial R_{ny}}{\partial y} + \frac{\partial R_{nz}}{\partial z}\right) - \left(\frac{\partial u g_x}{\partial x} + \frac{\partial v g_y}{\partial y} + \frac{\partial w g_z}{\partial z}\right) \\
& - L_v \epsilon \\
& - \left\{\frac{\partial(\tau_{xx}u + \tau_{yx}v + \tau_{zx}w)}{\partial x} + \frac{\partial(\tau_{xy}u + \tau_{yy}v + \tau_{zy}w)}{\partial y}\right. \\
& \left. + \frac{\partial(\tau_{xz}u + \tau_{yz}v + \tau_{zz}w)}{\partial z}\right\}
\end{aligned}
$$

All Together

- As with the derivation of the momentum equation, we can utilize Newtonian expressions to relate the velocity gradients and stresses, namely:

$$\tau_{xx} = -2\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu(\vec{\nabla}\cdot\vec{U}) \qquad \tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = -2\mu\frac{\partial v}{\partial y} + \frac{2}{3}\mu(\vec{\nabla}\cdot\vec{U}) \qquad \tau_{xz} = \tau_{zx} = -\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{zz} = -2\mu\frac{\partial w}{\partial z} + \frac{2}{3}\mu(\vec{\nabla}\cdot\vec{U}) \qquad \tau_{zy} = \tau_{yz} = -\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

Or more compactly:

$$\tau_{ij} = -2\mu S_{ij} + \frac{2}{3}\mu(\vec{\nabla}\cdot\vec{U})\delta_{ij}$$

Thermal Energy Equation

All Together

- We now subtract the E equation (slide 13) from the complete energy equation to yield the **thermal energy equation**:

$$\underbrace{\rho\frac{DI}{Dt}}_{1} = \underbrace{-\vec{\nabla}\cdot\vec{q}}_{2}\underbrace{-\vec{\nabla}\cdot(p\vec{U})}_{3}\underbrace{-\vec{\nabla}\cdot\overrightarrow{R_n}}_{4}+\underbrace{L_v\epsilon}_{5}+\underbrace{\mu\Phi_\nu}_{6}$$

Terms (per unit volume)

1. Rate of gain of internal energy
2. Rate of internal energy input by conduction
3. Reversible rate of internal energy increase by compression
4. Rate of internal energy input by net radiation
5. Rate of internal energy input by phase change
6. Irreversible rate of internal energy increase by viscous dissipation

Thermal Energy Equation

- The viscous term written out in full is:

$$\Phi_\nu = -2\mu S_{ij} S_{ij} + \frac{2}{3}\mu(\vec{\nabla} \cdot \vec{U})^2$$

- The thermal energy equation is written in terms of internal energy.
- However, we want to express the equation in terms of temperature and heat capacity.
- Let's work to relate I to T, θ and c_p, c_v.

Thermal Energy Equation

- Recall from thermodynamics that $I = I(\alpha, T)$, where α is the specific volume and T the absolute temperature.
- Thus,

$$dI = \left(\frac{\partial I}{\partial \alpha}\right)_T d\alpha + \left(\frac{\partial I}{\partial T}\right)_\alpha dT$$

- Multiplying by the density and considering the substantial derivatives:

$$\rho \frac{DI}{Dt} = \left(\frac{\partial I}{\partial \alpha}\right)_T \rho \frac{D\alpha}{Dt} + \rho c_v \frac{DT}{Dt}$$

where c_v is the specific heat of the fluid at constant volume.

Thermal Energy Equation

- Writing the specific volume as the inverse of the density and using the product rule of calculus:

$$\rho \frac{D\alpha}{Dt} = \rho \frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\rho}\vec{\nabla}\cdot\vec{U}$$

- Then for incompressible flow where $\vec{\nabla}\cdot\vec{U} = 0$ we have:

$$\rho c_v \frac{DT}{Dt} = -\vec{\nabla}\cdot\vec{q} - \vec{\nabla}\cdot\overrightarrow{R_n} + L_v\epsilon + \mu\Phi'_\nu$$

where the viscous dissipation term simplifies to:

$$\Phi_\nu = -2\mu S_{ij}S_{ij}$$

Thermal Energy Equation

- Using Fourier's Law of Heat Conduction allows us to write the second term as a function of temperature:

$$\vec{q} = -\kappa \vec{\nabla} T$$

where κ is the thermal conductivity of the fluid. Thus,

$$-\vec{\nabla} \cdot \vec{q} = \kappa \vec{\nabla} \cdot \vec{\nabla} T = \kappa \vec{\nabla}^2 T$$

- Substitution yields the thermal energy equation in terms of T:

$$\rho c_v \frac{DT}{Dt} = \kappa \vec{\nabla}^2 T - \vec{\nabla} \cdot \overrightarrow{R_n} + L_v \epsilon + \mu \Phi'_\nu$$

This equation yields temperature changes from heat conduction, radiation divergence, phase change, and viscous heating

Thermal Energy Equation

- For a constant pressure fluid we can make the following substitution:

$$dI = -p d\alpha + c_p dT$$

which for an incompressible fluid leads to

$$\rho c_p \frac{DT}{Dt} = \kappa \vec{\nabla}^2 T - \vec{\nabla} \cdot \overrightarrow{R_n} + L_v \epsilon + \mu \Phi'_\nu$$

which is essentially stating that we can switch between c_v and c_p, which is justified when the pressure terms are neglected in a gas flow energy equation.

Thermal Energy Equation

- The expression is approximately an enthalpy change:

$$\frac{DT}{Dt} = \frac{\kappa}{\rho c_p}\vec{\nabla}^2 T - \frac{1}{\rho c_p}\vec{\nabla}\cdot\overrightarrow{R_n} + \frac{L_v \epsilon}{\rho c_p} + \frac{\mu \Phi'_\nu}{\rho c_p}$$

- If viscous heating is small and we define thermal diffusivity as $K = \kappa / c_p T$:

$$\boxed{\frac{DT}{Dt} = K\vec{\nabla}^2 T - \frac{1}{\rho c_p}\vec{\nabla}\cdot\overrightarrow{R_n} + \frac{L_v \epsilon}{\rho c_p}}$$

or for potential temperature:

$$\boxed{\frac{D\theta}{Dt} = K\vec{\nabla}^2 \theta - \frac{1}{\rho c_p}\vec{\nabla}\cdot\overrightarrow{R_n} + \frac{L_v \epsilon}{\rho c_p}}$$

Summary

We have now derived the following basic equations of atmospheric dynamics, assumptions, approximations, and specific cases of their application:

- Conservation of momentum
- Assumptions and approximations to the momentum equation
- Ekman layer
- Taylor-Proudman Theorem
- Thermal wind
- Mechanical energy equation
- Thermal energy equation