# LES of Turbulent Flows: Lecture 23

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Fall 2016

# Overview

1. Lagrangian Particle Dispersion Modeling in LES

# Lagrangian Particle Dispersion Modeling in LES

- This is a special lecture on Lagrangian particle dispersion in LES created by Brian Bailey

# Lagrangian Particle Dispersion Modeling in LES

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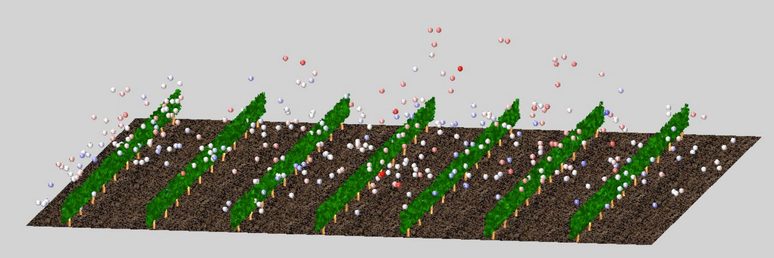
November 18, 2014

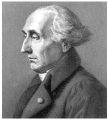
# Visualizations



# Lagrangian vs Eulerian Reference Frames

## Eulerian

## Lagrangian



Best for smoothly varying scalar fields (i.e., continuum)

Governing Equation

$$\frac{\partial C}{\partial t} + \frac{\partial u_j C}{\partial x_j} = D\frac{\partial^2 C}{\partial x_j x_j}$$

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# Lagrangian vs Eulerian Reference Frames

## Eulerian

Best for smoothly varying scalar fields (i.e., continuum)

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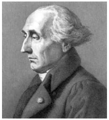
$$\frac{\partial C}{\partial t} + \frac{\partial u_j C}{\partial x_j} = D \frac{\partial^2 C}{\partial x_j x_j}$$

## Lagrangian



Best for discrete sources, or when details of individual particles are of interest

Governing Equation

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i$$

# Numerical solution

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i$$

$$\frac{x_i(t+\Delta t) - x_i(t)}{\Delta t} = u_i(t)$$

$$\boxed{x_i(t+\Delta t) = x_i(t) + u_i(t)\Delta t}$$

# Note

## Side Note:

This form assumes particles are massless.

Could add generic velocity (say $u_i^*$) to account for gravitational settling, inertia, etc.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i + u_i^*$$

# Numerical solution example

$$\boxed{x_i(t+\Delta t) = x_i(t) + u_i(t)\Delta t}$$

Consider $x(0) = 0$
$u(x=0) = 1$, $u(x=0.5) = 2$, $u(x=1) = 1.5$
$\Delta t = 0.1$

# Numerical solution example

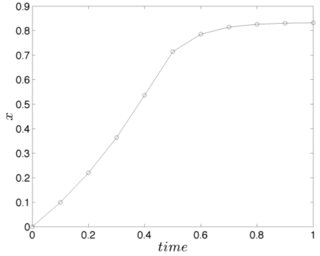
$$\boxed{x_i(t+\Delta t) = x_i(t) + u_i(t)\Delta t}$$

Consider $x(0) = 0$

$u(x=0) = 1,\ u(x=0.5) = 2,\ u(x=1) = 1.5$

$\Delta t = 0.1$

| $t$ | $x$ | $u$ |
|---|---|---|
| 0 | 0 | 1 |
| 0.1 | 0.1 | 1.2 |
| 0.2 | 0.22 | 1.44 |
| 0.3 | 0.36 | 1.73 |
| 0.4 | 0.54 | 0.71 |

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# Application to LES

What's the problem if we want to apply this to LES?

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$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i = \underbrace{\tilde{u}_i}_{\text{resolved}} + \underbrace{u_{s,i}}_{\text{subgrid}}$$

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# Application to LES

What's the problem if we want to apply this to LES?

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i = \underbrace{\tilde{u}_i}_{\text{resolved}} + \underbrace{u_{s,i}}_{\text{subgrid}}$$

We don't know $u_{s,i}$!

# Framework for modeling $u_{s,i}$

Could neglect it ($u_{s,i} = 0$)

e.g.,

**Pure Convection:**

Gopalakrishnan, S. G., and R. Avissar, 2000: An LES study of the impacts of land surface heterogeneity on dispersion in the convective boundary layer. *J. Atmos. Sci.*, **57**, 352–371.

**Near-Canopy Flow:**

Bailey, B. N., R. Stoll, E. R. Pardyjak, and W. F. Mahaffee, 2014: The effect of canopy architecture and the structure of turbulence on particle dispersion. *Atmos. Env.*, **95**, 480–489.

# Framework for modeling $u_{s,i}$

Modeling $u_{s,i}$:
where should we start?

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# Framework for modeling $u_{s,i}$

Modeling $u_{s,i}$:
where should we start?

Let's copy the RANS people.

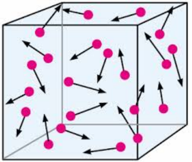
**Why?** RANS is essentially LES with the grid scale equal to the domain size....so this *should* be easier.

# Framework for modeling $u_{s,i}$

Lagrangian dispersion in RANS:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \underbrace{\overline{u}_i}_{\text{mean}} + \underbrace{u_i}_{\text{fluctuations}}$$

# RANS models



Analogy to molecular motion (Brownian motion):
**Langevin Equation**

$$\mathrm{d}u_i = \underbrace{-\boldsymbol{a} u_i \mathrm{d}t}_{\text{memory}} + \underbrace{\boldsymbol{b} \mathrm{d}\xi_i}_{\text{diffusion}}$$

$u_i$ - molecule velocity
$\mathrm{d}\xi_i$ - random Gaussian process with mean zero and variance $\mathrm{d}t$

# Langevin Equation

Application to isotropic turbulence:
$u_i \rightarrow$ Lagrangian particle velocity

$$\mathrm{d}u_i = -\boldsymbol{a} u_i \mathrm{d}t + \boldsymbol{b} \mathrm{d}\xi_i$$

How do we get $\boldsymbol{a}$ and $\boldsymbol{b}$?

# Langevin Equation: finding $b$

$$\mathrm{d}u_i = -\boldsymbol{a} u_i \mathrm{d}t + \boldsymbol{b} \mathrm{d}\xi_i \qquad (1)$$

$\boldsymbol{b}$ comes directly from Kolmogorov's second hypothesis

Lagrangian structure function:

$$D(\Delta t) = \langle (\Delta w)^2 \rangle = C_0 \varepsilon \Delta t$$

**Provided $\boldsymbol{\Delta} t$ is in the internal subrange (i.e., $\boldsymbol{\tau}_\eta \ll \Delta t \ll \tau_L$)**

# Langevin Equation: finding $b$

$$\mathrm{d}u_i = -\boldsymbol{a}u_i\mathrm{d}t + \boldsymbol{b}\mathrm{d}\xi_i \tag{1}$$

$\boldsymbol{b}$ comes directly from Kolmogorov's second hypothesis

Lagrangian structure function:

$$D(\Delta t) = \langle(\Delta w)^2\rangle = C_0\varepsilon\Delta t$$

**Provided $\boldsymbol{\Delta}t$ is in the internal subrange (i.e., $\boldsymbol{\tau}_\eta \ll \Delta t \ll \tau_L$)**

Square Eq.1 and take ensemble average:

$$\langle(\Delta w)^2\rangle = -\langle\boldsymbol{a}w^2\cancelto{\approx 0}{(\Delta t)^2}\rangle - \boldsymbol{a}\boldsymbol{b}\cancelto{0}{\langle w\Delta\xi\rangle}\Delta t + \boldsymbol{b}^2\cancelto{\Delta t}{\langle(\Delta\xi)^2\rangle}$$

$$\langle(\Delta w)^2\rangle = \boldsymbol{b}^2\Delta t = C_0\varepsilon\Delta t \rightarrow \boxed{\boldsymbol{b} = (C_0\varepsilon)^{-1/2}}$$

# Langevin Equation: finding $a$

$$\mathrm{d}u_i = -\boldsymbol{a}u_i\mathrm{d}t + \boldsymbol{b}\mathrm{d}\xi_i$$

Using stochastic calculus, we can solve this equation analytically

$$w(t) = w(0)e^{-\boldsymbol{a}t} + \boldsymbol{b}e^{-\boldsymbol{a}t}\int_0^t e^{\boldsymbol{a}s}\xi(s)\mathrm{d}s$$

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# Langevin Equation: finding $a$

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Square this equation and take ensemble average:

$$\langle w^2(t)\rangle = \langle w^2(0)\rangle e^{-2\boldsymbol{a}t} + \cancelto{0}{\langle w(0)\rangle} e^{-2\boldsymbol{a}t}\int_0^t e^{\boldsymbol{a}s}\xi(s)\mathrm{d}s + \langle \boldsymbol{b}^2 e^{-2\boldsymbol{a}t}\left[\int_0^t e^{\boldsymbol{a}s}\xi(s)\mathrm{d}s\right]^2\rangle$$

$$\langle w^2(t)\rangle = \langle w^2(0)\rangle e^{-2\boldsymbol{a}t} + \frac{\boldsymbol{b}^2}{2\boldsymbol{a}}\left[1 - e^{-2\boldsymbol{a}t}\right]$$

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# Langevin Equation: finding $a$

$$\langle w^2(t)\rangle = \langle w^2(0)\rangle e^{-2\boldsymbol{a}t} + \frac{\boldsymbol{b}^2}{2\boldsymbol{a}}\left[1 - e^{-2\boldsymbol{a}t}\right] \qquad (2)$$

For homogeneous and isotropic turbulence,

$$\langle w^2(t)\rangle = \langle w^2(0)\rangle = \sigma_w^2 \text{ (const.)}$$

Make this substitution and evaluate Eq. 2 at $t \to \infty$

$$\sigma_w^2 = \frac{\boldsymbol{b}^2}{2\boldsymbol{a}}$$

$$\boxed{\boldsymbol{a} = \frac{\boldsymbol{b}^2}{2\sigma_w^2} = \frac{C_0\varepsilon}{2\sigma_w^2}}$$

# Langevin Equation

Application to homogeneous isotropic turbulence

$$\mathrm{d}u_i = -\frac{C_0\varepsilon}{2\sigma^2}u_i\mathrm{d}t + \left(C_0\varepsilon\right)^{1/2}\mathrm{d}\xi_i$$

for homogeneous isotropic turbulence,

$\dfrac{2\sigma^2}{C_0\varepsilon} = \tau_L$ is the integral timescale

$$\mathrm{d}u_i = -\frac{u_i}{\tau_L}\mathrm{d}t + \left(C_0\varepsilon\right)^{1/2}\mathrm{d}\xi_i$$

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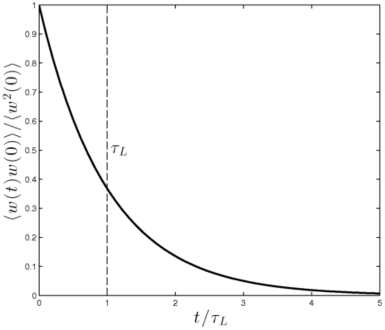
# Langevin Equation

$$\mathrm{d}w = \underbrace{-\frac{w}{\tau_L}\mathrm{d}t}_{\mathrm{I}} + \underbrace{(C_0\varepsilon)^{1/2}\,\mathrm{d}\xi_i}_{\mathrm{II}}$$

I Gives correct integral timescale of $\tau_L$ (long-time behavior)

$$\frac{\langle w(t)w(0)\rangle}{\langle w^2(0)\rangle} = e^{-t/\tau_L}$$

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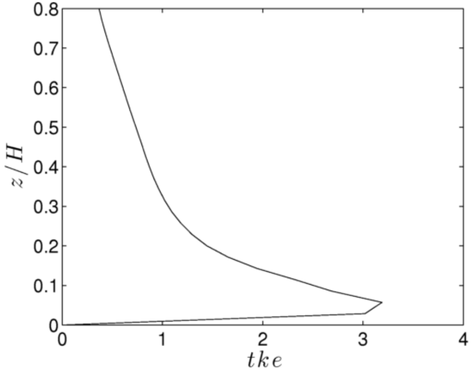
# Langevin Equation

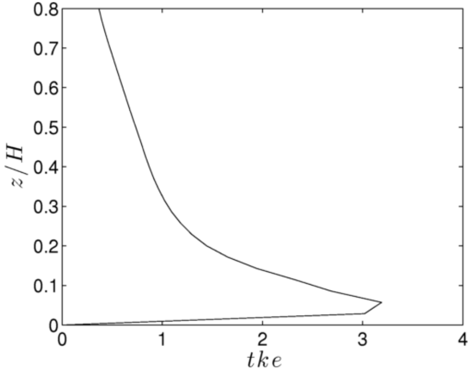
$$\mathrm{d}w = \underbrace{-\frac{w}{\tau_L}\mathrm{d}t}_{\mathrm{I}} + \underbrace{(C_0\varepsilon)^{1/2}\,\mathrm{d}\xi_i}_{\mathrm{II}}$$

- I Gives correct integral timescale of $\tau_L$ (long-time behavior)
- II Makes velocity consistent with Kolmogorov's second hypothesis (short-time behavior)

# Inhomogeneous Turbulence (in 1D)



Inhomogeneous Turbulence (in 1D)




$\frac{\partial k}{\partial z} \neq 0$ implies a mean flux!

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# Well-Mixed Condition

## Well-Mixed Condition[1] or Thermodynamic Constraint[2]

An initially well-mixed (uniform) particle distribution must remain well-mixed for all time in the absence of sources or sinks (second law of thermodynamics).

[1]Thomson, D. J., 1987: Criteria for the selection of stochastic models of particle trajectories in turbulent flows. *J. Fluid Mech.*, **180**, 529–556.

[2]Pope, S. B., 1987: Consistency conditions for random walk models of turbulent dispersion. *Phys. Fluids*, **30**, 2374–2379.

# Langevin Equation: Inhomogeneous Turbulence

$$\mathrm{d}u_i = \underbrace{\boldsymbol{a}_0\mathrm{d}t}_{\text{drift correction}} + \underbrace{\boldsymbol{a}_1 u_i \mathrm{d}t}_{\text{memory}} + \underbrace{\boldsymbol{b}\mathrm{d}\xi_i}_{\text{diffusion}}$$

# Langevin Equation: Inhomogeneous Turbulence

How to determine unknown coefficients?

**Fokker-Planck Equation**

$$\frac{\partial P_E}{\partial t} + \frac{\partial u_i P_E}{\partial x_i} = -\frac{\partial (a P_E)}{\partial u_i} + \frac{1}{2}\frac{\partial^2 (b^2 P_E)}{\partial u_i^2}$$

Advection-diffusion for Eulerian velocity PDF – Eulerian equivalent of Langevin equation.

For derivation see:

van Kampen, N.G.; 2nd ed., 1981. *Stochastic Processes in Physics and Chemistry*. North-Holland Pub. Co., 465 pp.
Rodean, H. C., 1996: *Stochastic Lagrangian Models of Turbulent Diffusion*. Amer. Meteor. Soc., Boston, MA, 84 pp.

# Langevin Equation: Inhomogeneous Turbulence

Solution in one dimension (unique):

$$\mathrm{d}w = \underbrace{\frac{1}{2}\frac{\partial \sigma_w^2}{\partial z}\mathrm{d}t}_{\mathrm{I}} - \underbrace{\left[\frac{C_0\varepsilon}{2\sigma_w^2} - \frac{w}{2\sigma_w^2}\frac{\partial \sigma_w^2}{\partial z}\right] w\mathrm{d}t}_{\mathrm{II}} + \underbrace{\left(C_0\varepsilon\right)^{1/2}\mathrm{d}\xi_i}_{\mathrm{III}}$$

- I Drift correction term
- II Memory term
- III Diffusion term

# Langevin Equation: Non-Uniqueness Problem

## Solution in three dimensions: method for determining Langevin coefficients is non-unique!

Thomson's (1987) 'simplest solution' (weak solution):

$$\mathrm{d}u_i = \frac{1}{2}\frac{\partial R_{il}}{\partial x_l}\mathrm{d}t - \frac{C_0\varepsilon}{2}R^{-1}_{ik}u_k + \frac{1}{2}\frac{\mathrm{d}R_{il}}{\mathrm{d}t}R^{-1}_{lj}u_j\mathrm{d}t + (C_0\varepsilon)^{1/2}\,\mathrm{d}\xi_i$$

$R_{ij}$ is the Reynolds stress tensor and $R^{-1}_{ij}$ is its inverse

We can add any arbitrary rotation vector to the drift term and we'll still satisfy the well-mixed condition.

# Langevin Equation: Rogue Trajectory Problem

$$\mathrm{d}w = \frac{1}{2}\frac{\partial \sigma_w^2}{\partial z}\mathrm{d}t - \left[\frac{C_0\varepsilon}{2\sigma_w^2} - \frac{w}{2\sigma_w^2}\frac{\partial \sigma_w^2}{\partial z}\right] w\mathrm{d}t + \left(C_0\varepsilon\right)^{1/2}\mathrm{d}\xi_i$$

It is possible for our Langevin equation to become unstable and get cases where $u_i \to \infty$

# Langevin Equation: Rogue Trajectory Problem

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It is possible for our Langevin equation to become unstable and get cases where $u_i \to \infty$

ROGUE TRAJECTORY!

SSSSH! This is our dirty little secret.

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# Rogue Trajectories

What can we do about rogue trajectories?

- *ad hoc* constraints (violates well-mixed condition)
- Yee and Wilson (2007): semi-analytical scheme
- Postma et al. (2012): refine timestep
- Bailey et al. (2014): semi-implicit scheme

# Application to LES

$$\mathrm{d}w_s = \frac{1}{2}\frac{\partial \sigma_s^2}{\partial z}\mathrm{d}t - \left[\frac{C_0\varepsilon_s}{2\sigma_s^2} - \frac{w_s}{2\sigma_s^2}\frac{\partial \sigma_s^2}{\partial z}\right] w_s \mathrm{d}t + \left(C_0\varepsilon_s\right)^{1/2}\mathrm{d}\xi_i^*$$

Replace 'fluctuating' quantities with subgrid quantities

- $w \rightarrow w_s$
- $\sigma^2 \rightarrow \sigma_s^2$
- $\varepsilon \rightarrow \varepsilon_s$ (for $\Delta$ in inertial subrange, $\overline{\varepsilon} \approx \overline{\varepsilon_s} = -\tilde{S}_{ij}\tau_{ij}$)

*NOTE: this form assumes horizontal homogeneity and that $\tau_{ij}$ is isotropic. See Weil et al. (2004) for fully general version.

# Langevin Equation: LES

e.g.,

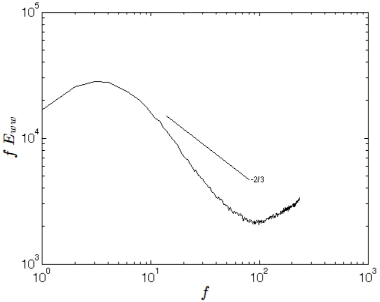

- Kemp, J. R. and Thomson, D. J. (1996). Dispersion in stable boundary layers using large-eddy simulation. Atmos. Env. 30:2911-2923.
- Weil, J. C. and Sullivan, P. P. and Patton, E. G. (2004). The use of large-eddy simulations in Lagrangian particle dispersion models. *J. Atmos. Sci.* 61:2877-2997.
- Vinkovic, I., Aguirre, C., and Simoëns, S. (2006). Large-eddy simulation and Lagrangian stochastic modeling of passive scalar dispersion in a turbulent boundary layer. J. Turb. 7:N30.


End Current Literature

(this is state-of-the-art)

# LES Lagrangian Energy Spectra

No SGS model ($u_{si} = 0$)



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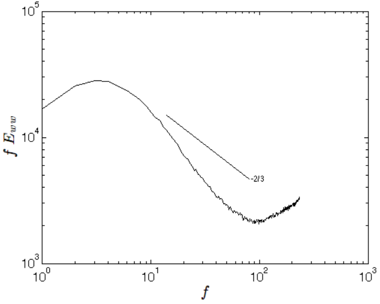
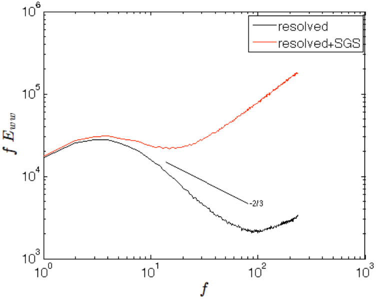



SGS model



# LES Lagrangian Energy Spectra

Where might all this energy be coming from?

1. Langevin equation is inappropriate?
2. Langevin coefficients are incorrect?
3. Rogue trajectories?

# Rogue Trajectories

## Where do ROGUE TRAJECTORIES come from?

Homogeneous version (1D RANS):

$$\mathrm{d}w = -\underbrace{\frac{C_0\varepsilon}{2\sigma^2}}_{1/\tau_L} w\mathrm{d}t + (C_0\varepsilon)^{1/2}\,\mathrm{d}\xi$$

Where do ROGUE TRAJECTORIES come from?

Homogeneous version (1D RANS):

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Inhomogeneous version (1D RANS):

$$\mathrm{d}w = \underbrace{\frac{1}{2}\frac{\partial\sigma^2}{\partial z}}_{\substack{\text{drift}\\\text{correction}}} - \underbrace{\left[\frac{C_0\varepsilon}{2\sigma^2} - \frac{w}{2\sigma^2}\frac{\partial\sigma^2}{\partial z}\right]}_{1/\tau_L} w\mathrm{d}t + (C_0\varepsilon)^{1/2}\,\mathrm{d}\xi$$

Where do ROGUE TRAJECTORIES come from?

Memory term:

$$-\underbrace{\left[\frac{C_0\varepsilon}{2\sigma_s^2}-\frac{w}{2\sigma_w^2}\frac{\partial\sigma_w^2}{\partial z}\right]}_{\frac{1}{\tau}=\frac{1}{\tau_1}-\frac{1}{\tau_2}} w\mathrm{d}t$$

- $\boldsymbol{\tau}_1$: Local decorrelation time scale (isotropic)
- $\boldsymbol{\tau}_2$: Heterogeneity decorrelation time scale

# Rogue Trajectories

$$-\underbrace{\left[\frac{C_0\varepsilon}{2\sigma_w^2}-\frac{w}{2\sigma_w^2}\frac{\partial\sigma_w^2}{\partial z}\right]}_{\frac{1}{\tau}}w\mathrm{d}t$$

What if $\tau$ turns out to be NEGATIVE? Or

$$C_0\varepsilon < w\frac{\partial\sigma_w^2}{\partial z}$$

Recall our autocorrelation function:

$$\frac{\langle w(t)w(0)\rangle}{\langle w^2(0)\rangle}=e^{-t/\tau}$$

# Rogue Trajectories

$$-\underbrace{\left[\frac{C_0\varepsilon}{2\sigma_w^2}-\frac{w}{2\sigma_w^2}\frac{\partial\sigma_w^2}{\partial z}\right]}_{\displaystyle\frac{1}{\tau}} w\mathrm{d}t$$

What could cause $\tau$ to be NEGATIVE?

$\Delta t$ not in the inertial subrange i.e., $\tau_L \lesssim \Delta t$

Thus $\dfrac{2\sigma_w^2}{C_0\varepsilon}$ is not the proper decorrelation timescale!

In this case, it is the problem not the discretization scheme that is unstable!!!!

# Rogue Trajectories

Generalizing to 3D (assume $\tau_{ij}$ is isotropic)

$$\mathrm{d}u_{s,i} = \frac{1}{2}\frac{\partial \sigma_s^2}{\partial x_i}\mathrm{d}t - \underbrace{\left(\frac{C_0\varepsilon_s}{2\sigma_s^2} - \frac{1}{2\sigma_s^2}\frac{\mathrm{d}\sigma_s^2}{\mathrm{d}t}\right)}_{1/\tau} u_{s,i}\mathrm{d}t + \left(C_0\varepsilon_s\right)^{1/2}\mathrm{d}\xi_i$$

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Generalizing to 3D (assume $\tau_{ij}$ is isotropic)

$$\mathrm{d}u_{s,i} = \frac{1}{2}\frac{\partial \sigma_s^2}{\partial x_i}\mathrm{d}t - \underbrace{\left(\frac{C_0\varepsilon_s}{2\sigma_s^2} - \frac{1}{2\sigma_s^2}\frac{\mathrm{d}\sigma_s^2}{\mathrm{d}t}\right)}_{1/\tau} u_{s,i}\mathrm{d}t + \left(C_0\varepsilon_s\right)^{1/2}\mathrm{d}\xi_i$$

Unstable if

$C_0\varepsilon_s < \dfrac{\mathrm{d}\sigma_s^2}{\mathrm{d}t}$ (this means $\tau$ is negative)

# Rogue Trajectories

Generalizing to 3D (anisotropic $\tau_{ij}$)

$$\mathrm{d}u_{s,i} = \frac{1}{2}\frac{\partial \tau_{il}}{\partial x_l}\mathrm{d}t - \frac{C_0\varepsilon_s}{2}\lambda_{ik}u_{s,k} + \frac{1}{2}\frac{\mathrm{d}\tau_{il}}{\mathrm{d}t}\lambda_{lj}u_{s,j}\mathrm{d}t + \left(C_0\varepsilon_s\right)^{1/2}\mathrm{d}\xi_i$$

Generalizing to 3D (anisotropic $\tau_{ij}$)

$$\mathrm{d}u_{s,i} = \frac{1}{2}\frac{\partial \tau_{il}}{\partial x_l}\mathrm{d}t - \frac{C_0\varepsilon_s}{2}\lambda_{ik}u_{s,k} + \frac{1}{2}\frac{\mathrm{d}\tau_{il}}{\mathrm{d}t}\lambda_{lj}u_{s,j}\mathrm{d}t + \left(C_0\varepsilon_s\right)^{1/2}\mathrm{d}\xi_i$$

Unstable if

$$G_{ij} = \delta_{ij} + \frac{\Delta t}{2}\left(-C_0\varepsilon_s\lambda_{ij} + \frac{\mathrm{d}\tau_{il}}{\mathrm{d}t}\lambda_{lj}\right)$$

$\boxed{|\lambda_{\text{max}}| > 1}$ ($\lambda_{\text{max}}$ is largest eigenvalue of $G_{ij}$)

# LES: Rogue Trajectory Problem

Possible Solution: Reduce $\Delta t$

Sometimes not computationally feasible.

# LES: Rogue Trajectory Problem

Possible Solution: *ad-hoc* intervention

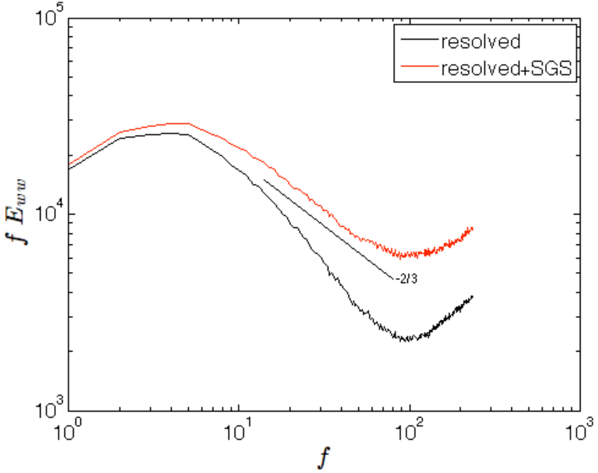
Violates well-mixed condition.

# LES: Rogue Trajectory Problem

Possible Solution: Use mean quantities to calculate memory term

$$\mathrm{d}u_{s,i} = \frac{1}{2}\frac{\partial \sigma_s^2}{\partial x_i}\mathrm{d}t - \left[\frac{C_0\overline{\varepsilon_s}}{2\overline{\sigma_s^2}} - \frac{u_{s,j}}{2\overline{\sigma_s^2}}\overline{\frac{\partial \sigma_s^2}{\partial x_j}}\right] u_{s,i}\mathrm{d}t + \left(C_0\varepsilon_s\right)^{1/2}\mathrm{d}\xi_i$$

# LES Energy Spectra



# LES: Rogue Trajectory Problem

## Possible Solution: Directly calculate $\tau_{L,s}$

We use Lagrangian scale-dependent SGS momentum model, which gives $\tau_{Ls}$

See:

Stoll, R., and Porté-Agel, F. (2006). Dynamic Subgrid-Scale Models for Momentum and Scalar Fluxes in Large-Eddy Simulations of Neutrally Stratified Atmospheric Boundary Layers Over Heterogeneous Terrain. *Water Resour. Res.* 42:W01409.

$$\mathrm{d}u_{s,i} = \frac{1}{2}\frac{\partial \sigma_s^2}{\partial x_i}\mathrm{d}t - \frac{u_{s,i}}{\tau_{Ls}}\mathrm{d}t + \left(C_0 \varepsilon_s\right)^{1/2}\mathrm{d}\xi_i$$

this form is unconditionally stable!