LES of Turbulent Flows: Lecture 16

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Overview

Mixed Models

- Both the similarity and nonlinear models exhibit a high level of correlation in *a priori* tests with measured values of τ_{ij}^{Δ}, but they underestimate the average dissipation and are numerically unstable
- Typically they are combined with an eddy-viscosity model to provide the proper level of dissipation.

Mixed Models

- An example is (Bardina et al. 1980)

$$\boxed{\tau_{ij} = C_L \left(\overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\ \overline{\widetilde{u}_j} \right) - 2(C_S \Delta)^2 |\widetilde{S}| \widetilde{S}_{ij}}$$

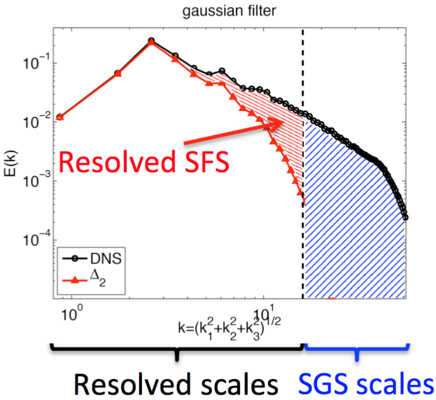
- The similarity term has a high level of correlation with τ_{ij}^{Δ} and the eddy-viscosity term provides the appropriate level of dissipation.

Mixed Models

- Proper justification for the mixed model did not exist at first but a more unified theory has developed in the form of approximate deconvolution or filter reconstruction modeling (Guerts pg 200, Sagaut pg 210)
- Idea: a SGS model should be built from 2 parts

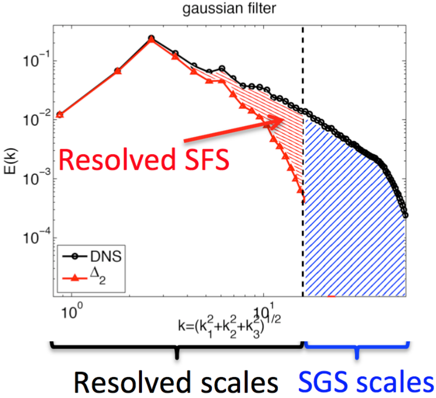
Mixed Models

- The **first part** accounts for the effect of the filter through an approximate reconstruction of the filter's effect on the velocity field (note the similarity model is a zero-order filter reconstruction).
- This is the model for the resolved SFS.



Mixed Models

- The **second part** accounts for the SGS component of τ_{ij}
- We then assume that we can build τ_{ij} as a linear combination of these two model components.



Mixed Models

A few last notes on Similarity models

- Bardina's model is exactly zero for a spectral cutoff filter
- Liu et al. form of the similarity model also fails. This is credited to the nonlocal structure of the cutoff filter. It breaks the central assumption of the similarity model – that the locally τ_{ij} decomposed at different levels is self similar

Modulated Gradient Model

- A related model of similar form to the nonlinear model is the Modulated Gradient Model (see Lu et al. 2008 and Lu and Porté-Agel 2010)
- The goal is to improve the magnitude of the τ_{ij} estimates while keeping the high level of correlation observed for nonlinear (gradient) models

Modulated Gradient Model

- Assume
$$\tau_{ij} = \widetilde{k}_r C_{ij}$$
where for consistency $C_{kk} = 2$
- Using our resolved stress $L_{ij} = \overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\,\overline{\widetilde{u}_j}$ and the Germano identity (more later), we can show that approximately
$$C_{ij} = 2(L_{ij}/L_{kk}) \Rightarrow \tau_{ij} = 2\widetilde{k}_r(L_{ij}/L_{kk})$$

Modulated Gradient Model

- This model suffers from some drawbacks, such as insufficient dissipation at high Re and that it is not material frame indifferent
- The authors suggested an improvement by replacing L_{ij} with $\widetilde{A}_{ik}$

$$\tau_{ij} = 2\widetilde{k}_r(\widetilde{A}_{ij}\widetilde{A}_{kk})$$

Note: we can also use

$$\widetilde{G}_{ij} = \frac{\Delta_x^2}{12}\frac{\partial \widetilde{u}_i}{\partial x}\frac{\partial \widetilde{u}_j}{\partial x} + \frac{\Delta_y^2}{12}\frac{\partial \widetilde{u}_i}{\partial y}\frac{\partial \widetilde{u}_j}{\partial y} + \frac{\Delta_z^2}{12}\frac{\partial \widetilde{u}_i}{\partial z}\frac{\partial \widetilde{u}_j}{\partial z}$$

Dynamic SGS Models

- So far we have given a general description of some commonly used SGS models
- **All of these models include at least one model coefficient** that must be prescribed, either based on theory with a specific set of assumptions (usually isotropy), from experimental data, or chosen ad hoc to get the "correct" *a posteriori* results from simulations
- Germano et al. (1991) developed a procedure to dynamically calculate these unknown model coefficients
- For scalars and compressible flow see Moin et al., PofF, (1991)

Dynamic SGS Models

- Recall: applying a low-pass filter to the N-S equations with a filter of characteristic width Δ (denoted by $\widetilde{\ }$) results in the unknown SFS stress term:

$$\boxed{\tau_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j} \tag{1}$$

- This term must be modeled with an SGS model to close our equation set

Dynamic SGS Models

- We can apply another filter (referred to as a test filter) to the filtered N-S equations at a larger scale (say 2Δ) denoted by a bar (-):

$$\overline{\frac{\partial \widetilde{u}_i}{\partial t}} + \overline{\frac{\partial \widetilde{u}_i \widetilde{u}_j}{\partial x_j}} = -\overline{\frac{\partial \widetilde{p}}{\partial x_i}} + \overline{\frac{1}{\mathrm{Re}} \frac{\partial^2 \widetilde{u}_i}{\partial x_j^2}} - \overline{\frac{\partial \tau_{ij}}{\partial x_j}} + F_i$$

- Our LES filter properties (commutation with differentiation) allows us to rewrite in standard filtered form

$$\frac{\partial \overline{\widetilde{u}_i}}{\partial t} + \frac{\partial \overline{\widetilde{u}_i \widetilde{u}_j}}{\partial x_j} = -\frac{\partial \overline{\widetilde{p}}}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 \overline{\widetilde{u}_i}}{\partial x_j^2} - \frac{\partial \overline{\tau}_{ij}}{\partial x_j} + F_i$$

- The convective term can be reformatted into our standard format using the same method we used for the original filtered LES equations (see Lecture 6)

$$
\begin{aligned}
\frac{\partial \overline{\widetilde{u}_i \widetilde{u}_j}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j} + \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j}\right) \\
&= \underbrace{\frac{\partial \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j}}{\partial x_j}}_{\text{I}} + \underbrace{\frac{\partial\left(\overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j}\right)}{\partial x_j}}_{\text{II}}
\end{aligned}
$$

- Term I is our standard form and we can move Term II to the RHS of our expression

Dynamic SGS Models

- Rearranging yields

$$\frac{\partial \overline{\widetilde{u}_i}}{\partial t} + \frac{\partial \overline{\widetilde{u}_i}\,\overline{\widetilde{u}_j}}{\partial x_j} = -\frac{\partial \overline{\widetilde{p}}}{\partial x_i} + \frac{1}{\mathsf{Re}}\frac{\partial^2 \overline{\widetilde{u}_i}}{\partial x_j^2} - \frac{\partial \overline{\tau}_{ij}}{\partial x_j} - \frac{\partial\left(\overline{\widetilde{u}_i\widetilde{u}_j} - \overline{\widetilde{u}_i}\,\overline{\widetilde{u}_j}\right)}{\partial x_j} + F_i$$

Note: $\overline{\tau_{ij}} = \overline{\widetilde{u_i u_j}} - \overline{\widetilde{u}_i\widetilde{u}_j}$

$$\frac{\partial \overline{\widetilde{u}_i}}{\partial t} + \frac{\partial \overline{\widetilde{u}_i}\,\overline{\widetilde{u}_j}}{\partial x_j} = -\frac{\partial \overline{\widetilde{p}}}{\partial x_i} + \frac{1}{\mathsf{Re}}\frac{\partial^2 \overline{\widetilde{u}_i}}{\partial x_j^2} - \frac{\partial\left(\overline{\widetilde{u_i u_j}} - \cancel{\overline{\widetilde{u}_i\widetilde{u}_j}} - \cancel{\overline{\widetilde{u}_i\widetilde{u}_j}} + \overline{\widetilde{u}_i}\,\overline{\widetilde{u}_j}\right)}{\partial x_j} + F_i$$

Dynamic SGS Models

- We can now write the SFS stress at the 2Δ level as:

$$\boxed{T_{ij} = \overline{\widetilde{u_i u_j}} - \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j}} \tag{2}$$

which leads to

$$\frac{\partial \overline{\widetilde{u}_i}}{\partial t} + \frac{\partial \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j}}{\partial x_j} = -\frac{\partial \overline{\widetilde{p}}}{\partial x_i} + \frac{1}{\mathsf{Re}}\frac{\partial^2 \overline{\widetilde{u}_i}}{\partial x_j^2} - \frac{\partial T_{ij}}{\partial x_j} + F_i$$

- We can also consider the stress at the smallest resolved scales (the Leonard stress we discussed in Lecture 13)

$$\boxed{L_{ij} = \overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j}} \tag{3}$$

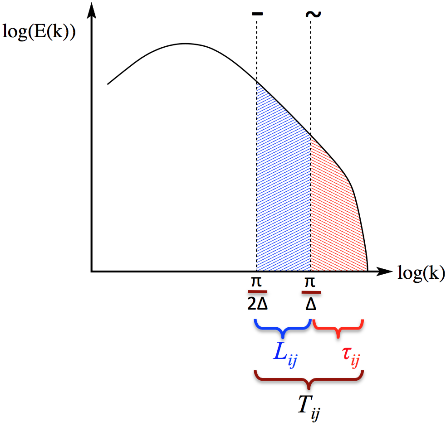
Dynamic SGS Models

- We can algebraically combine Equations (1), (2), (3) as

$$L_{ij} = T_{ij} - \overline{\tau_{ij}} \tag{4}$$

$$\overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\ \overline{\widetilde{u}_j} = \overline{\widetilde{u_i u_j}} - \overline{\widetilde{u}_i}\ \overline{\widetilde{u}_j} - \overline{\widetilde{u_i u_j}} + \overline{\widetilde{u}_i \widetilde{u}_j}$$

- Graphically, this looks like



Dynamic SGS Models

- Equation (4) is an identity – **it is exact!** It can be exploited to derive model coefficients for common SGS models
- This identity is usually referred to as the **Germano identity**
- We will use the Smagorinsky model as an example of how one can use the Germano identity to find model coefficients.
- Procedurally, we can follow the same steps (presented next) for any base SGS model

Dynamic Smagorinsky Model

- The first assumption we must make is that the same model (e.g. Smagorinsky model) can be applied for the stress at Δ and $\alpha\Delta$ (say, 2Δ)
- Using the Smagorinsky model in the Germano identity (Note Smagorinsky is only for anisotropic part)

$$L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} = T_{ij} - \overline{\tau_{ij}}$$

$$\overline{\widetilde{u}_i\widetilde{u}_j} - \overline{\widetilde{u}_i}\,\overline{\widetilde{u}_j} = -2(C_S\alpha\Delta)^2\left|\overline{\widetilde{S}}\right|\overline{\widetilde{S}}_{ij} + \overline{2(C_S\Delta)^2\left|\widetilde{S}\right|\widetilde{S}_{ij}}$$

Dynamic Smagorinsky Model

- For the next parts we will follow Lilly (1992)
- We can rearrange this equation to write an expression for the error associated with using the Smagorinsky model in the Germano identity

$$e_{ij} = L_{ij} - \frac{1}{3}\delta_{ij} L_{kk} - \left[-2(C_S \alpha \Delta)^2 \left| \overline{\widetilde{S}} \right| \overline{\widetilde{S}}_{ij} + \overline{2(C_S \Delta)^2 \left| \widetilde{S} \right| \widetilde{S}_{ij}} \right]$$

- This can be rewritten as (note: we assume L_{ij} is trace free)

$$e_{ij} = L_{ij} - C_S^2 M_{ij}$$

where

$$M_{ij} = 2\Delta^2 \left[\overline{\left| \widetilde{S} \right| \widetilde{S}_{ij}} - \alpha^2 \left| \overline{\widetilde{S}} \right| \overline{\widetilde{S}}_{ij} \right]$$

- Problem: This is 9 equations with only 1 unknown!

Dynamic Smagorinsky Model

- Lilly (1992) proposed to minimize the error in a least-squares sense. That is, we want the least-square error of using the Smagorinsky model in the Germano identity
- The squared error is

$$e_{ij}^2 = (L_{ij} - C_S^2 M_{ij})^2 = L_{ij}^2 - 2C_S^2 L_{ij} M_{ij} + (C_S^2)^2 M_{ij} M_{ij}$$

- We want the minimum w.r.t. C_S^2 (*i.e.*, $\partial e_{ij}^2 / \partial C_S^2 = 0$)

$$\frac{\partial e_{ij}^2}{\partial C_S^2} = -2L_{ij} M_{ij} + 2C_S^2 M_{ij} M_{ij} = 0$$

Solving for C_S^2 yields

$$\boxed{C_S^2 = \frac{L_{ij} M_{ij}}{M_{ij} M_{ij}}}$$

Dynamic Smagorinsky Model

- Problem: the above local form of the dynamic Smagorinsky coefficient is numerically unstable
- Remember that the instantaneously energy cascade can be forward or backwards
- In simulations, this was found to lead to numerical instability (having $\pm C_S^2$)
- The instability is attributed to high time correlations of C_S^2 (*i.e.*, when C_S^2 is negative at a point it tends to stay negative)
- Why do we have this problem? **We had to make 2 assumptions to derive the C_S^2 equation!**

Assumptions in the Dynamic Model

1st Assumption

- C_S^2 is constant over the filter width $\alpha\Delta$ (– filter in the equations)
- Recall our basic definition of a convolution filter

$$\tilde{\phi}(\vec{x},t) = \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}$$

  If we look at the error equations, we notice that C_S^2 falls under the bar filter

$$e_{ij} = L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} - \left[-2(C_S\alpha\Delta)^2 \left|\overline{\widetilde{S}}\right| \overline{\widetilde{S}}_{ij} + \overline{2(C_S\Delta)^2 \left|\widetilde{S}\right| \widetilde{S}_{ij}}\right]$$

- This is actually a **set of integral equations** if we don't make our assumption!

Assumptions in the Dynamic Model

1^{st} **Assumption**, continued

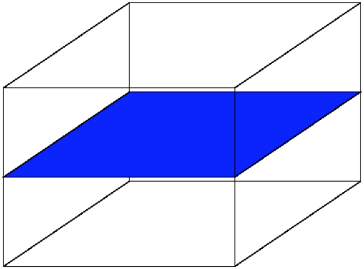
- Ghosal et al. (1995) solved this equation for C_S^2 everywhere using a variational method – which is **very expensive and complex**
- The constant C_S^2 (w.r.t. the test filter) assumption contributes to the previously discussed numerical instability
- The typical method (instead of the Ghosal method) is to enforce the Germano identity in an average sense

$$\boxed{C_S^2 = \frac{\langle L_{ij} M_{ij} \rangle}{\langle M_{ij} M_{ij} \rangle}}$$

Assumptions in the Dynamic Model

1st Assumption, continued

- Constraining C_S^2 removes its oscillations – resulting in stable simulations
- Typically, the average is enforced over some region of spatial homogeneity
- For example in a homogeneous boundary layer over horizontal planes
- $\langle\ \rangle$ is an averaging operator, *e.g.*, C_S^2 varies only in the wall normal direction



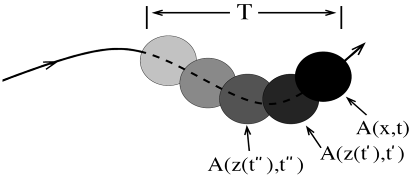
Assumptions in the Dynamic Model

1^{st} **Assumption**, continued

- Meneveau et al. (1996) developed the Lagrangian Dynamic model based on the idea that the Germano identity should be enforced along fluid particle trajectories
- Using 1^{st}-order time and space estimates, the average of any quantity A (*e.g.*, L_{ij}) can be defined as

$$\langle A(\vec{x})\rangle^{n+1} = \gamma\left[A(\vec{x})\right]^{n+1} + (1-\gamma)\left[A(\vec{x}-\vec{u}^n\Delta t)\right]^n$$

where $\gamma \equiv (\Delta t/T^n)/(1+\Delta t/T^n)$ and T is the Lagrangian timescale that controls how far back in time the average goes



2nd Assumption

- When we applied the Smagorinsky model to the Germano identity at 2 different scales, **we made the assumption that the same C_S^2 applies at both scales**
- *i.e.*, we assumed $C_S^2(\Delta) = C_S^2(2\Delta)$, or in other words scale invariance of C_S^2
- This assumption is not bad provided that both of our filter scales Δ and 2Δ are in the inertial subrange of turbulence
- We will violate this assumption in some region of the flow (*e.g.*, near the wall in a boundary layer when $z \leq 2\Delta$) for cases with at least 1 direction of flow anisotropy

Assumptions in the Dynamic Model

2nd Assumption, continued

- Porté-Agel et al., (2000) developed a generalized dynamic model where C_S^2 is a function of scale
- They made the weaker assumption that C_S^2 follows a power law distribution at the smallest resolved scales, *e.g.*

$$\frac{C_S^2(2\Delta)}{C_S^2(\Delta} = \frac{C_S^2(4\Delta)}{C_S^2(2\Delta}$$

Assumptions in the Dynamic Model

2nd **Assumption**, continued

- So now in our equation for C_S^2, we have

$$M_{ij} = 2\Delta^2 \left[\overline{\left| \widetilde{S} \right| \widetilde{S}_{ij}} - \alpha^2 \beta \left| \overline{\widetilde{S}} \right| \overline{\widetilde{S}}_{ij} \right]$$

where

$$\beta = \frac{C_S^2(2\Delta)}{C_S^2(\Delta)}$$

is the scale-dependence coefficient