LES of Turbulent Flows: Lecture 14

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Overview

1. Recap: Smagorinsky Model
2. Two-Equation Eddy-Viscosity Models
3. One-Equation Eddy-Viscosity Models
4. Two-Point Eddy-Viscosity Models

Recap: Eddy-Viscosity Models

- In Lecture 12 we said that eddy-viscosity models are of the form:

$$\text{momentum} \qquad \tau_{ij} = -2\nu_T \widetilde{S}_{ij}$$

$$\text{scalars} \qquad q_i = -D_T \frac{\partial \widetilde{\theta}}{\partial x_i}$$

where

$$D_T = \frac{\nu_T}{\mathsf{Pr}},$$

- $\widetilde{S}_{ij}$ is the filtered strain rate
- ν_T is eddy-viscosity
- D_T is eddy-diffusivity
- Pr is the SGS Prandtl number

Recap: Smagorinsky Model

$$\nu_T = (C_S \Delta)^2 |\widetilde{S}|$$

- $\Delta = (\Delta_x \Delta_y \Delta_z)^{\frac{1}{3}}$ is the effective grid scale (see Deardorff 1970 or Scotti et al, 1993)
- $C_S \Delta$ is the length scale – squared for dimensional consistency
- $|\widetilde{S}| = \sqrt{2\widetilde{S}_{ij}\widetilde{S}_{ij}}$ is the magnitude of the filtered strain rate tensor with units of $[T^{-1}]$. It serves as part of the velocity scale – think $\partial\langle u\rangle/\partial z$ in Prandtl's theory

Recap: Smagorinsky Model

- The final model is

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\nu_T \widetilde{S}_{ij} = -2(C_S\Delta)^2|\widetilde{S}|\widetilde{S}_{ij}$$

- In order to close the model, we need a value of C_S (usually called the Smagorinsky or Smagorinsky-Lilly coefficient)
- In Lecture 13 we derived an expression for the Smagorinsky coefficient and showed that $C_S \approx 0.17$ (for a spectral cutoff filter)

Recap: Smagorinsky Model

- Recall that dimensionally,

$$\nu_T = \left[\frac{L^2}{T}\right] \sim U\ell$$

- Smagorinsky model prescribed both U and ℓ
- Subsequent models attempted to improve adaptation of the eddy-viscosity approach to local flow properties
- One-equation models prescribe ℓ, while U is predicted by the flow
- In two-equation models, both ℓ and U are predicted by the flow

Two-Equation Eddy-Viscosity Models

- Deardorff (1973): model the evolution of τ_{ij} (Sagaut pg. 243)

$$\frac{\partial \tau_{ij}}{\partial t} = -\frac{\partial(\bar{u}_k \tau_{ij})}{\partial x_k} - \tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k}$$

$$\underbrace{-\frac{\partial(\overline{u'_i u'_j u'_k})}{\partial x_k}}_{\text{I}} + \underbrace{\overline{p'\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)}}_{\text{II}}$$

$$\underbrace{-\frac{\partial(\overline{u'_i p'})}{\partial x_j}}_{\text{III}} - \underbrace{\frac{\partial(\overline{u'_j p'})}{\partial x_i}}_{\text{IV}} - \underbrace{2\nu\overline{\left(\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}\right)}}_{\text{V}}$$

- where subgrid kinetic energy $k_r = \tau_{kk}/2$
- The terms in this equation have to be modeled.

Two-Equation Eddy-Viscosity Models

- I (tripple correlation)

$$\overline{u_i' u_j' u_k'} = -C_{3m} f(k_r, \tau_{ij})$$

- II (pressure-strain correlation)

$$p' \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)} = -C_m f(k_r, \tau_{ij})$$

- III, IV (pressure-velocity correlations) were ignored
- V (dissipation)

$$2\nu \overline{\left(\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \right)} = C_e f(k_r)$$

- where $C_m = 4.13$, $C_e = 0.7$, and $C_{3m} = 0.2$

Two-Equation Eddy-Viscosity Models

- To close the model, a separate equation for subgrid turbulence kinetic energy (k_r) is required.
- This model is similar to the a 2^{nd}-order RANS closure (see Speziale 1991 for a review of RANS closures including 2^{nd}-order models)
- The need for these additional equations add significantly to the complexity and computational cost of the simulation
- Despite the added complexity, the models are still subject to the limitations of the basic eddy-viscosity approach

One-Equation Eddy-Viscosity Models

- Deardorff (1980) suggested a simpler 1-equation approach to avoid the need to solve prognostic equations for τ_{ij} (in addition to N-S equations)
- This model is equivalent to a k-ℓ 1-equation RANS model (see Speziale, 1991)
- This was also proposed earlier for the isotropic part of a 2-part eddy-viscosity model by Schumann (1975).

One-Equation Eddy-Viscosity Models

- However, the model is usually credited to Deardorff (especially in the atmospheric community)
- Models based on D80 are popular in the atmospheric boundary layer due to the ability to include SGS transport or energy drain effects as extra parameters in the SGS TKE equation (*e.g.* for SGS canopy drag, buoyancy forces, etc.)

One-Equation Eddy-Viscosity Models

- In this model, ℓ is prescribed and U is predicted by the flow
- Assume that $U \sim \sqrt{k_r}$
- Thus, eddy-viscosity is modeled as

$$\nu_T = C_1 \ell_d \sqrt{k_r}$$

- Here, $C_1 (\sim 0.1)$ is the "Deardorff coefficient"
- $C_1 \ell_d$ is ℓ (similar to Smagorinsky) and $\sqrt{k_r}$ is U
- Thus, the model is given by

$$\tau_{ij} = -2 C_1 \ell_d \sqrt{k_r} \widetilde{S}_{ij}$$

- In order to close this model, we must prescribe ℓ_d and determine k_r from a separate transport equation.

One-Equation Eddy-Viscosity Models

- The parameterized dimensional form of the k_r transport equation according to D80 is given by:

$$\frac{\partial k_r}{\partial t} = -\frac{\partial \tilde{u}_j k_r}{\partial x_j} + 2\nu_T S_{ij} S_{ij} - D_T \frac{\partial \tilde{b}}{\partial z} + \frac{\partial}{\partial x_j} 2\nu_T \frac{\partial k_r}{\partial x_j} - \epsilon$$

  where, $\tilde{b} = (g/\theta_o)\tilde{\theta}$ is buoyancy, θ_o is a constant reference potential temperature, and $\tilde{\theta}$ is potential temperature

- To complete this transport equation and close the model, we need expressions for ν_T, D_T, and ϵ

One-Equation Eddy-Viscosity Models

- According to D80

$$\nu_T = C_1 \ell_d \sqrt{k_r}$$

$$D_T = \left(1 + 2\frac{\ell_d}{\Delta}\right)\nu_T$$

$$\epsilon = C_e \frac{k_r^{3/2}}{\ell_d}$$

where

$$C_e = f_c\left(0.19 + 0.51\frac{\ell_d}{\Delta}\right)$$

and

$$f_c = 1 + \frac{2}{\left(\frac{z_w}{\Delta z_w} + 1.5\right)^2 - 3.3}$$

One-Equation Eddy-Viscosity Models

- A note from D80 about f_c, which is used to enhance near-wall dissipation (in our notation):

> *Close to the surface, however, C_e was increased by a 'wall-effect' factor of up to 3.9 to prevent k_r from becoming unduly large there*

- Some later studies (*e.g.* Moeng 1984) merely set $C_e = 3.9$ at the first model level.
- Others follow that of Nieuwstadt (1990) by using f_c
- Your instructor believes the latter is more physically meaningful and in the spirit of Deardorff's comment.

One-Equation Eddy-Viscosity Models

- Finally, we have to prescribe ℓ_d

$$\ell_d = \begin{cases} \Delta & \dfrac{\partial \tilde{b}}{\partial z} \leq 0 \\ \min\left[\Delta, \dfrac{1}{2}\dfrac{\sqrt{k_r}}{N}\right] & \dfrac{\partial \tilde{b}}{\partial z} > 0 \end{cases}$$

  where $N = \sqrt{\partial \tilde{b}/\partial z}$ is the Brunt-Väisälä frequency

- D80 tied the notion of the grid scale to static stability (*i.e.*, as static stability increases, the characteristic length scale should reduce to less than the effective grid spacing)

One-Equation Eddy-Viscosity Models

- The complete D80 model in dimensional form:

$$
\begin{aligned}
\tau_{ij} &= -2C_1 \ell_d \sqrt{k_r}\widetilde{S}_{ij} \\
\frac{\partial k_r}{\partial t} &= -\frac{\partial \tilde{u}_j k_r}{\partial x_j} + 2\nu_T S_{ij} S_{ij} - D_T \frac{\partial \tilde{b}}{\partial z} + \frac{\partial}{\partial x_j} 2\nu_T \frac{\partial k_r}{\partial x_j} - \epsilon \\
\nu_T &= C_1 \ell_d \sqrt{k_r} \\
D_T &= \left(1 + 2\frac{\ell_d}{\Delta}\right)\nu_T \\
\epsilon &= C_e \frac{k_r^{3/2}}{\ell_d} \\
\ell_d &= \Delta(\partial\tilde{b}/\partial x_3 \leq 0), \min\left[\Delta, 0.5\sqrt{k_r}/N\right](\partial\tilde{b}/\partial x_3 > 0)
\end{aligned}
$$

One-Equation Eddy-Viscosity Models

- Another form of the transport equation is given in dimensionless form (see Geurts pg 227, Debliquy 2001)

$$\frac{\partial k_r}{\partial t} = -\frac{\partial \tilde{u}_j k_r}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{C_2}{C_1}\Delta\sqrt{k_r}\frac{\partial k_r}{\partial x_j}\right) + \frac{1}{\mathrm{Re}}\frac{\partial^2 k_r}{\partial x_j^2} + \Pi - \epsilon$$

- viscous dissipation is typically modeled based on isotropy assumption $\epsilon = (C_k/\Delta)k_r^{3/2}$
- C_2 is a constant on the order of unity and $C_k \approx 1.7$ is the Kolmogorov "constant"
- Note: the transport terms for pressure and SGS kinetic energy have been modeled by the 2nd-term on the RHS of the equation

One-Equation Eddy-Viscosity Models

Gibbs and Fedorovich (2016) addressed issues w/ D80

- Does the closure require the stability-dependent turbulence length-scale?
- The Prandtl number formulation can lead to stability patchiness
- Is there a need for near-wall enhancement of dissipation anymore?

One-Equation Eddy-Viscosity Models

- To address the first point, GF16 eliminates the stability dependence of the length scale assuming that the grid spacing is small enough to match, at least approximately, the reduced turbulence length-scale?

$$\ell_d = \Delta$$

One-Equation Eddy-Viscosity Models

- To address the second point, GF16 creates a new formulation of the Prandtl number ($\mathrm{Pr} = \nu_T / D_T$)
- Using the D80 formulations

$$\mathrm{Pr} = \left(1 + 2\frac{\ell_d}{\Delta}\right)^{-1}$$

which means based on the stability criteria

$$\mathrm{Pr} = \begin{cases} 1/3 & \frac{\partial \tilde{b}}{\partial z} \leq 0 \\ \to 1 \text{ as } \ell_d \to 0 & \frac{\partial \tilde{b}}{\partial z} > 0 \end{cases}$$

- This means that under stable conditions, with $\ell_d < \Delta$, ν_T is persistently smaller than D_T, while most studies (*e.g.*, Ohya 2001, Grachev et al. 2007, Zilitinkevich et al. 2012) show that at least with moderate stability Pr≈ 1.

One-Equation Eddy-Viscosity Models

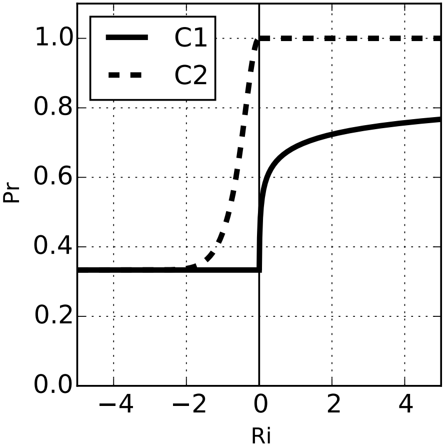
- In addition, GF16 reported that the use of the D80 approach sometimes led to problematic patchiness of the resolved fields when adjoining cells were of opposite stability.
- The new formulation is:

$$\mathsf{Pr} = \begin{cases} \left(3 - 2e^{-\mathrm{Ri}^2}\right)^{-1} & \frac{\partial \tilde{b}}{\partial z} \leq 0 \\ 1 & \frac{\partial \tilde{b}}{\partial z} > 0 \end{cases}$$

where Ri is the gradient Richardson number

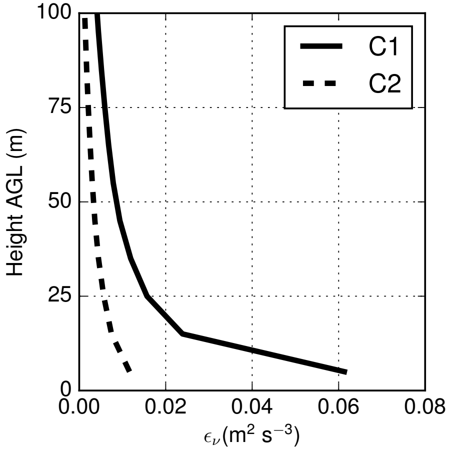
$$\mathrm{Ri} = \frac{\dfrac{\partial \tilde{b}}{\partial z}}{\left(\dfrac{\partial \tilde{u}}{\partial z}\right)^2 + \left(\dfrac{\partial \tilde{v}}{\partial z}\right)^2}$$

- C_1 is D80, C_2 is GF16
- The outcome is that the relative effect of mechanical mixing is enhanced with stable stratification compared to the original formulation of D80



One-Equation Eddy-Viscosity Models

- Finally, GF16 suggests that near-wall enhancement of dissipation is unnecessary, provided that the grid spacing is adequately small.
- This argument follows the findings in Moeng et al (2007), which effectively corresponds to setting $f_c = 1$

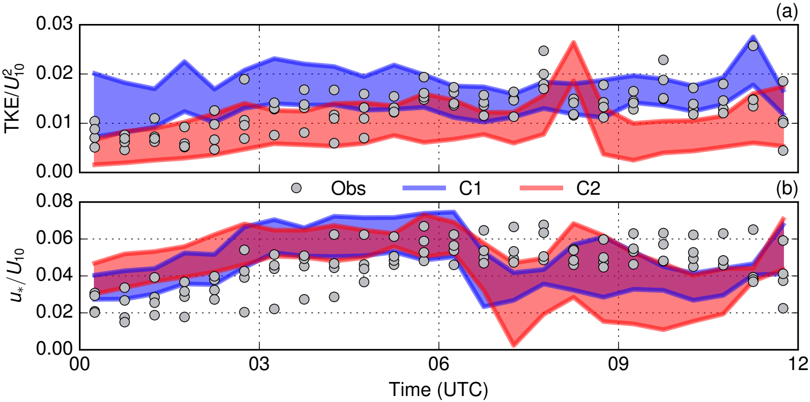


One-Equation Eddy-Viscosity Models

- The GF16 modifications to D80 are summarized here

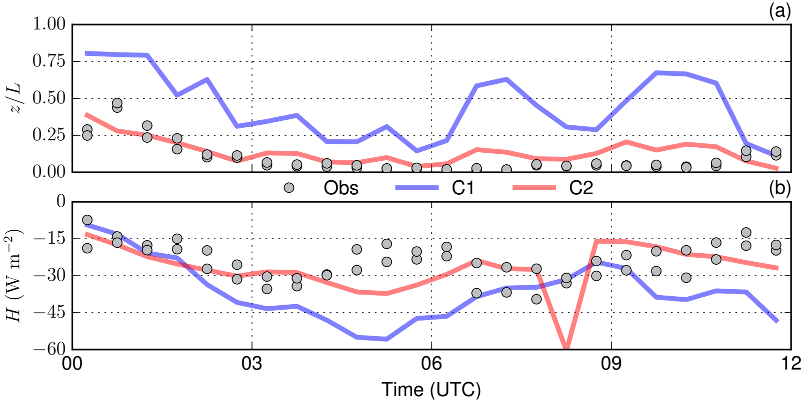
$$
\begin{aligned}
\nu_T &= C_1 \Delta \sqrt{k_r}\ , \\
D_T &= \begin{cases} \left(3 - 2e^{-\mathrm{Ri}^2}\right)\nu_T & \frac{\partial \tilde{b}}{\partial z} \leq 0 \\ \nu_T & \frac{\partial \tilde{b}}{\partial z} > 0 \end{cases}, \\
\epsilon &= 0.7 \frac{E^{\frac{3}{2}}}{\Delta}\ .
\end{aligned}
$$

One-Equation Eddy-Viscosity Models



- New formulation seems to better capture near-surface TKE until flow becomes really stable ($t \approx 8\mathrm{h}$)
- Both formulations seem to overpredict u_* under moderate stability and overpredict when the flow becomes more stable.

One-Equation Eddy-Viscosity Models



- New formulation seems to better capture near-surface stability of the flow
- New formulation seems to better capture the near-surface sensible heat flux

One-Equation Eddy-Viscosity Models

Gibbs and Fedorovich (2016) has weak points

- Run for a single case
- Grid spacing of $10\ \mathrm{m}$ might be too coarse to meet the assumptions of the modifications
- The LES code used was driven (initialized and nudged in time) by a large-scale weather model (WRF), which means the formulations may be sensitive to model bias
- The LES code uses low-order advection schemes

Two-Point Eddy-Viscosity Models

- Based on Metais and Lesieur, 1992
- See Sagaut pg 124 or Lesieur et al., 2005 "Large-Eddy Simulation of Turbulence"
- This model is an attempt to go beyond the Smagorinsky model while keeping, in physical space, the same scaling as the spectral eddy-viscosity model of Kriachnan (1976)

Two-Point Eddy-Viscosity Models

- Idea: in physical space build an eddy-viscosity normalized by

$$\sqrt{\frac{E_{\vec{x}}(k_c)}{k_c}} \qquad \text{with} \qquad k_c = \frac{\pi}{\Delta}$$

  and where $E_{\vec{x}}$ is the local kinetic energy spectrum at point $\vec{x}$

Two-Point Eddy-Viscosity Models

- $E_{\vec{x}}$ must be evaluated in terms of physical space quantities. The best candidate for this is the 2nd-order structure function:

$$F^{\mathsf{iso}}(r) = \left\langle [\vec{u}(\vec{x},t) - \vec{u}(\vec{x}+\vec{r},t)]^2 \right\rangle$$

- Note: the isotropic 2nd-order structure function spectrum (Fourier transform) is equivalent to the the Kolmogorov $k^{-5/3}$ energy spectrum
- See Pope sections 6.2 and 6.4, Wyngaard (2010), Gibbs and Fedorovich (2016b) for the relationship between structure functions and energy spectrum

Two-Point Eddy-Viscosity Models

- For the two-point eddy-viscosity model, the local structure function is used

$$F_2(\vec{x}, \Delta) = \left\langle \left[\widetilde{\vec{u}}(\vec{x}, t) - \widetilde{\vec{u}}(\vec{x} + \vec{r}, t) \right]^2 \right\rangle_{||\vec{r}||}$$

  where we now have a local statistical average over the nearest 6 points (or 4 points in a boundary layer).

- Assuming a $k^{-5/3}$ energy spectrum from 0 to k_c, we get

$$\boxed{\nu_T(\vec{x}, \Delta, t) = 0.0105 C_k^{-3/2} \Delta \sqrt{F_2(\vec{x}, \Delta)}}$$

  where C_k is the Kolmogorov "constant"

Two-Point Eddy-Viscosity Models

- If we replace the velocity increments by 1st-order spatial derivatives, we can show that

$$\nu_T \approx 0.777(C_S\Delta)^2\sqrt{2\widetilde{S}_{ij}\widetilde{S}_{ij} - \tilde{\omega}_i\tilde{\omega}_i}$$

  where $\tilde{\omega} = \vec{\nabla} \times \widetilde{\vec{u}}$ is the filtered vorticity.
- We can imagine that the two-point (or structure function) model is the Smagorinsky model in a strain/vorticity formulation