LES of Turbulent Flows: Lecture 5

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Overview

1. Recap of N-S equations
2. Approximating the equations of motion
3. Pros and cons of each method
4. Scale separation

Recap

- The Navier-Stokes equations describe the motion of fluids, through the conservation of mass, momentum, and energy.
- The equations are nonlinear, which complicates our ability to analyze and simulate fluid flows.
- Why? The nonlinearity creates a continuous spectrum of different flow features.

Recap of N-S equations

- This spectrum contains very large integral scales and very small dissipation scales.
- The simultaneous representation of both large and small scales makes for a very large computational problem.
- Current computational resources limit the amount of small features that can be accurately simulated.

Recap

- The complexity of a flow can be reduced to alleviate this computation bottleneck.
- This technique is aimed at capturing the primary features of a flow with sufficient detail and accepting that the full turbulent solution may not be obtained perfectly.
- This sets the stage for LES as a tool to solve for the "reduced" flow.
- Before diving into LES, we will go over the description of the DNS, LES, and RANS techniques.

Approximating the equations of motion

- Numerical studies require that the equations of motion for a (compressible, incompressible, Boussinesq) fluid must be approximated on a computational grid using discrete physical points or basis functions.
- Three basic methodologies are prevalent in turbulence application and research:
    - Direct Numerical Simulation (DNS)
    - Large-Eddy Simulation (LES)
    - Reynolds-Averaged Navier-Stokes (RANS)

Approximating the equations of motion

Direct Numerical Simulation

- The DNS approach focuses on finding a numerically-accurate solution to the N-S equations (*i.e.*, resolve all eddies).
- As we saw last class, it is an expensive operation.

Approximating the equations of motion

Large-Eddy Simulation

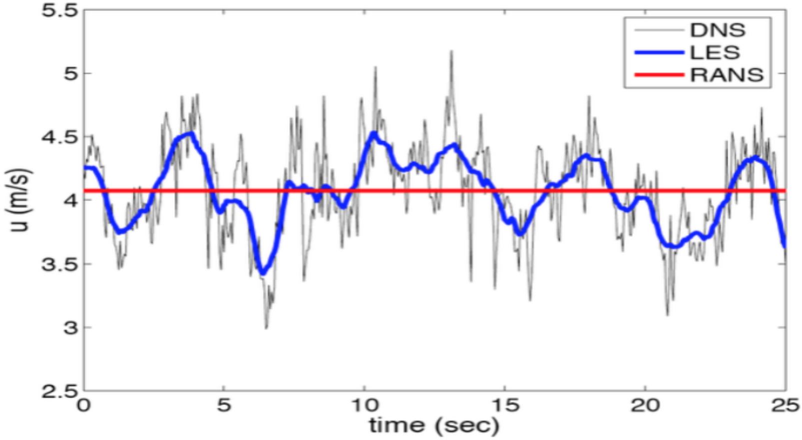
- The LES approach emphasizes capturing those primary flow features that are larger than a prescribed filter width (Δ) (*i.e.*, resolve larger-eddies and model smaller "universal" ones).
- Since Δ is prescribed, one has control over the required resolution and computational effort.
- The LES approach introduces the closure problem and reduces the information of the resolved flow.
- Has primarily trended toward engineering applications, but its use in atmospheric science is increasing.

Approximating the equations of motion

Reynolds-Averaged Navier-Stokes

- The RANS approach focuses on a statistical description of the basic fluctuation-correlations (*i.e.*, model ensemble statistics).
- Can be used to study flows with realistic complexity.

Approximating the equations of motion



Pros and cons of each method

Direct Numerical Simulation

- Pros
  - Does not require the use of a turbulence model
  - Accuracy is only limited by computational capabilities since errors are generally only due to sensitivity to perturbations or accumulated round-off errors.
  - All aspects of the flow in time and space are available, which is not possible for experiments (*i.e.*, 3D velocity and scalar fields).

Pros and cons of each method

Direct Numerical Simulation

- Cons
  - Restricted to low-Re flows with relatively simple geometries.
  - Very high memory and computational time costs.
  - Typically the "largest possible" number of grid points is used without proper convergence evaluation (*i.e.*, does not allow for systematic variation of numerical and physical parameters)

Pros and cons of each method

Large-Eddy Simulation

- Pros
  - Only small scales require require modeling. Since the small scales are likely insensitive to specifics of the flow, these models can be rather simple and "universal".
  - Much cheaper computational cost than DNS.
  - Unsteady predictions of the flow are made, which implies that information about extreme events are available over some period of time.
  - Properly designed LES should allow for $Re \Rightarrow \infty$.
  - In principle, we can gain as much accuracy as desired by refining our numerical grid.

Pros and cons of each method

Large-Eddy Simulation

- Cons
  - Basic assumption (small scales are universal) requires independence of small (unresolved) scales from boundary conditions (especially important for flow geometry). This is a problem for boundary layers, where proximity to the wall defines some of the smallest scales of the flow – which necessitates explicit modeling of the region.
  - Still very costly for many practical engineering applications.
  - Filtering and turbulence theory of small scales still needs development for complex geometry and highly anisotropic flows.

Pros and cons of each method

Reynolds-Averaged Navier-Stokes

- Pros
  - Low computational cost (can obtain mean statistics in a short time).
  - Can be used for highly complex flow geometries.
  - When combined with empirical information, can be highly useful for engineering applications and to parameterize near-wall behavior.

Pros and cons of each method

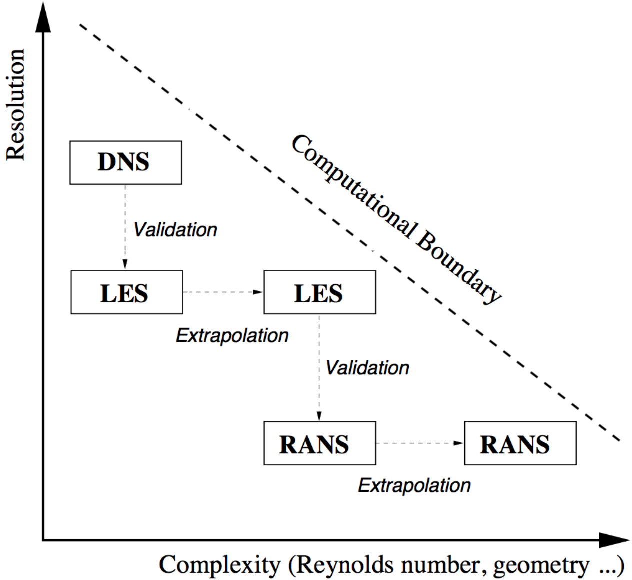
Reynolds-Averaged Navier-Stokes

- Cons
  - Only steady flow phenomena can be explored with full advantage of computational reduction.
  - Models are not universal since dynamic consequences of all scales must be represented. Usually a pragmatic "tuning" of model parameters is required for specific applications.
  - Most accurate turbulence models give rise to highly complex equations sets, which can lead to numerical formulation and convergence issues.

Relationship between each method

- DNS delivers the most accurate data (in general).
- DNS can be used to validate and analyze aspects of LES, such as numerical methods and subgrid model.
- LES provides a more complete picture of turbulent flow than RANS.
- RANS can be validated against LES data (where LES is used to obtain statistical information about the flow).

Relationship between each method



Scale separation

- So far, we have discussed LES very generically:
  - Resolve only the largest energy-containing scales.
  - Model the small "universal" scales.
- We will now formalize the idea of scale separation mathematically to show how to deal with the equations of motion and derive subgrid models.

Scale separation

- How do we accomplish scale separation?
  - A **low-pass filter**
- What is meant by *low-pass*?
  - A low-pass filter *passes* over signals with a frequency (wavenumber) *lower* than a certain cutoff frequency (wavenumber) and only smooths signals with a frequency (wavenumber) higher than the cutoff value.
- Our goal for the low-pass filter:
  - Attenuate (smooth) **high frequency** (high wavenumber/small scale) turbulence features that are smaller than a prescribed characteristic scale Δ while leaving **low-frequency** (low wavenumber/large scale) motions unchanged.

Filtering (Sagaut Chapter 2, Pope Chapter 13.2)

- The formal (mathematical) LES filter is a convolution filter defined for a quantity $\phi(\vec{x}, t)$ in physical space as:

$$\tilde{\phi}(\vec{x}, t) = \int_{-\infty}^{\infty} \phi(\vec{x} - \vec{\zeta}, t) G(\vec{\zeta}) d\vec{\zeta}$$

- $G \equiv$ the convolution kernel of the chosen filter.
- G is associated with the characteristic cutoff scale Δ (also called the filter width).

Convolution

- So, we have the convolution filter

$$\tilde{\phi}(\vec{x},t) = \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}$$

- Here we will use Pope's notation for the Fourier transform:

$$F\{\phi(x)\} = \int_{-\infty}^{\infty} e^{-ikx}\phi(x)dx$$

Convolution

- Take the Fourier transform of $\tilde{\phi}(\vec{x})$ (dropping t for simplicity):

$$F\{\tilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{x}} \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}d\vec{x}$$

- We can define a new variable $\vec{r} = \vec{x} - \vec{\zeta}$:

$$\begin{aligned} F\{\tilde{\phi}(\vec{x})\} &= \int_{-\infty}^{\infty} e^{-i\vec{k}(\vec{r}+\vec{\zeta})} \int_{-\infty}^{\infty} \phi(\vec{r})G(\vec{\zeta})d\vec{\zeta}d\vec{r} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\vec{k}(\vec{r}+\vec{\zeta})}\phi(\vec{r})G(\vec{\zeta})d\vec{\zeta}d\vec{r} \end{aligned}$$

Note: $d\vec{r} = d\vec{x}$ because $\vec{\zeta} \neq f(\vec{x})$ and we changed the order of integration

- We left off with:

$$F\{\tilde{\phi}(\vec{x})\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\vec{k}(\vec{r}+\vec{\zeta})}\phi(\vec{r})G(\vec{\zeta})d\vec{\zeta}d\vec{r}$$

- Recall that $e^{a+b} = e^a e^b$:

$$\begin{aligned} F\{\tilde{\phi}(\vec{x})\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\vec{k}\vec{r}}e^{-i\vec{k}\vec{\zeta}}\phi(\vec{r})d\vec{r}G(\vec{\zeta})d\vec{\zeta} \\ &= \int_{-\infty}^{\infty} e^{-i\vec{k}\vec{r}}\phi(\vec{r})d\vec{r}\int_{-\infty}^{\infty} e^{-i\vec{k}\vec{\zeta}}G(\vec{\zeta})d\vec{\zeta} \\ &\boxed{= F\{\phi(\vec{x})\}F\{G(\vec{\zeta})\}} \end{aligned}$$

where we changed the order for integration

Convolution

- We found this convolution relationship:

$$\boxed{F\{\tilde{\phi}(\vec{x})\} = F\{\phi(\vec{x})\}F\{G(\vec{\zeta})\}}$$

- Segaut writes this as:

$$\tilde{\hat{\phi}}(\vec{k},\omega) = \hat{\phi}(\vec{k},\omega)\hat{G}(\vec{k}\omega)$$

where $(\hat{\ })$ denotes a Fourier coefficient.

Convolution

- $F\{\tilde{\phi}(\vec{x})\} = F\{\phi(\vec{x})\}F\{G(\vec{\zeta})\}$ or $\hat{\tilde{\phi}}(\vec{k},\omega) = \hat{\phi}(\vec{k},\omega)\hat{G}(\vec{k}\omega)$
- $\hat{G}$ is the transfer function associated with the filter kernel G.
- Recall, that a transfer function is the wavespace (Fourier) relationship between the input and output of a linear system.
- A convolution is an integral that expresses the amount of overlap of one function G as it is shifted over another function ϕ (*i.e.*, it blends one function with another).

Decomposition into resolved and subfilter components

- Just as G is associated with a filter scale Δ (filter width), $\hat{G}$ is associated with a cutoff wavenumber k_c.
- In a similar manner to Reynolds decomposition, we can use the filter function to decompose the velocity field into resolved and unresolved (or subfilter) components:

$$\underbrace{\phi(\vec{x},t)}_{\text{total}} = \underbrace{\tilde{\phi}(\vec{x},t)}_{\text{resolved}} + \underbrace{\phi'(\vec{x},t)}_{\text{subfilter}}$$

Fundamental properties of “proper” LES filters

- The filter should not change the value of a constant a

$$\int_{-\infty}^{\infty} G(\vec{x})d\vec{x} = 1 \Rightarrow \tilde{a} = a$$

- Linearity

$$\widetilde{\phi + \zeta} = \tilde{\phi} + \tilde{\zeta}$$

(this is automatically satisfied for a convolution filter)

- Commutation with differentiation

$$\widetilde{\frac{\partial \phi}{\partial \vec{x}}} = \frac{\partial \tilde{\phi}}{\partial \vec{x}}$$

LES and Reynolds Operators

- In the general case, LES filters that verify these properties are not Reynolds operators.
- Recall for a Reynolds operator (average) defined by $\langle \ \rangle$
  - $\langle a\phi \rangle = a\langle \phi \rangle$
  - $\langle \phi' \rangle = 0$
  - $\langle \phi + \zeta \rangle = \langle \phi \rangle + \langle \zeta \rangle$
  - $\langle \langle \phi \rangle \rangle = \langle \phi \rangle$
  - $\langle \dfrac{\partial \phi}{\partial \vec{x}} \rangle = \dfrac{\partial \langle \phi \rangle}{\partial \vec{x}}$

LES and Reynolds Operators

- For our LES filter, in general using Sagaut's shorthand $\int_{-\infty}^{\infty} \phi(\vec{r},t)G(\vec{\zeta})d\vec{\zeta} = G \star \phi$:
  - $\tilde{\tilde{\phi}} = G \star G \star \phi = G^2 \star \phi \neq \tilde{\phi} = G \star \phi$
  - $\tilde{\phi}' = G \star (\phi - G \star \phi) \neq 0$
- For an LES filter, a twice filtered variable is not equal to a single filtered variable – unlike it is for a Reynolds average.
- Likewise, the filtered subfilter scale component is not equal to zero as it is for a Reynolds average.

Common (or classic) LES filters

- Box or "top-hat" filter (equivalent to a local average):

$$\underbrace{G(r) = \begin{cases} \frac{1}{\Delta} & \text{if } r \leq \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}}_{\text{filter function}} \qquad \underbrace{\hat{G}(k) = \frac{\sin\left(k\Delta/2\right)}{k\Delta/2}}_{\text{transfer function}}$$

- Gaussian filter (γ typically 6):

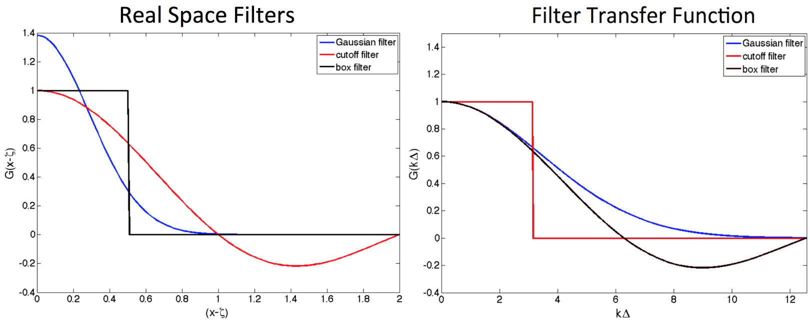
$$G(r) = \left(\frac{\gamma}{\pi\Delta^2}\right)^{\frac{1}{2}} \exp\left(\frac{-\gamma|r|^2}{\Delta^2}\right) \qquad \hat{G}(k) = \exp\left(\frac{-\Delta^2 k^2}{4\gamma}\right)$$

- Spectral or sharp-cutoff filter:

$$G(r) = \frac{\sin(k_c r)}{k_c r} \qquad \hat{G}(k) = \begin{cases} 1 & \text{if } |k| \leq k_c \\ 0 & \text{otherwise} \end{cases}$$

recall that k_c is the characteristic wavenumber cutoff.

Common (or classic) LES filters



Only the Gaussian filter is local in both real and wave space.

Filters: local vs non-local

- Where we apply a filter is important.
- The Fourier transform of a box filter is a wave, and the inverse transform of a spectral cutoff filter is a wave.
- This means we will get different results for these two filters depending on where they are applied.

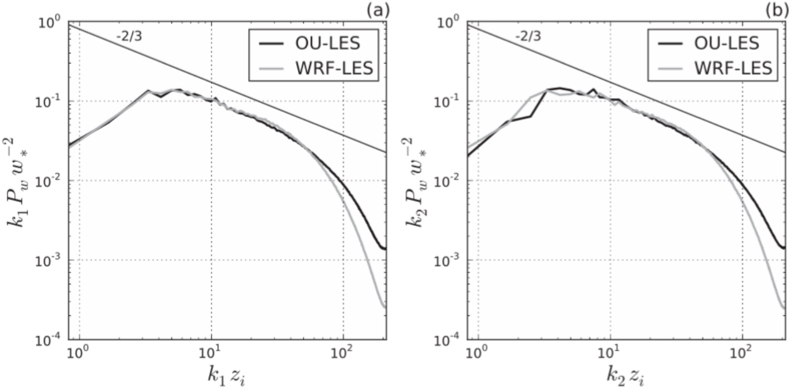
Filters: local vs non-local

- Thus, we say a box filter is local in physical space and non-local in wavespace.
- Conversely, a cutoff filter is local in wavespace and non-local in physical space.
- When a filter is non-local, think about it in terms of adding "wiggles" everywhere.
- As opposed to the box and cutoff filters, the Gaussian filter is (semi) local in both physical space and wavespace (semi because it differs by constants).
- This is because the Fourier transform of a Gaussian is also a Gaussian.

Spectral resolution

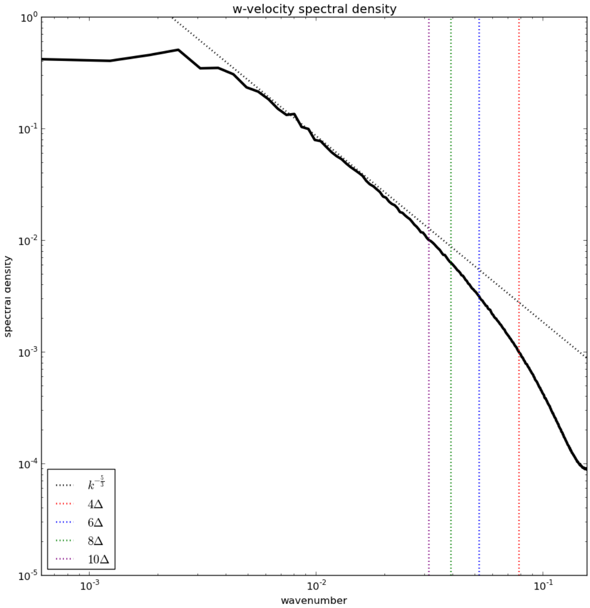
- We can tie the notion of filters and local/non-local behavior to numerical models and resolution.
- The notion of “spectral” or “effective” resolution arises because the spectra from LES often does not fall at Δ, but rather at some larger scale that is a multiple of Δ.
- For instance, a finite difference scheme (perhaps 2$^{\text{nd}}$-order central difference) is local in physical space, but non-local in wavespace.
- This impacts smaller wavenumbers (larger scales).

Spectral resolution



From Gibbs and Fedorovich (2014).

Spectral resolution



Convolution example

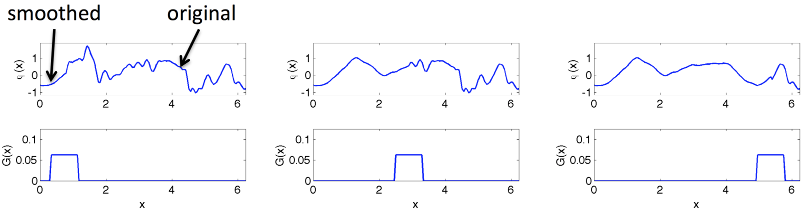
- We defined the convolution of two functions as:

$$\tilde{\phi}(\vec{x},t) = \int_{-\infty}^{\infty} \phi(\vec{x}-\vec{\zeta},t)G(\vec{\zeta})d\vec{\zeta}$$

- How can we interpret this relation? G, our filter kernel, 'moves' along our other function ϕ and smooths it out (provided it is a low-pass filter).

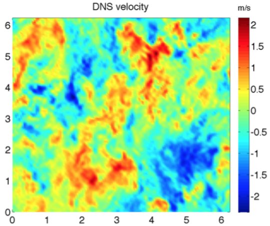
Convolution example

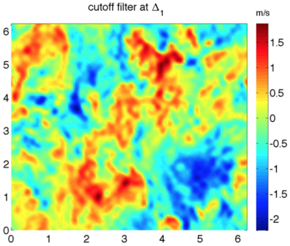
- One example is using a box filter applied in real space.
- See convolution_example.m (and associated iso_vel128.mat data file) on Canvas or website.

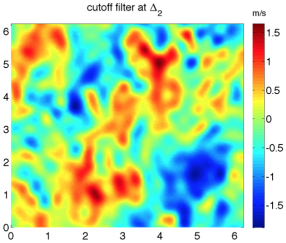


Filtering turbulence (real space, cutoff filter)

Note: here (and throughout the presentation) we are using **DNS data from Lu et al.** (International Journal of Modern Physics C, 2008).







Filtering turbulence (real space, Gaussian filter)

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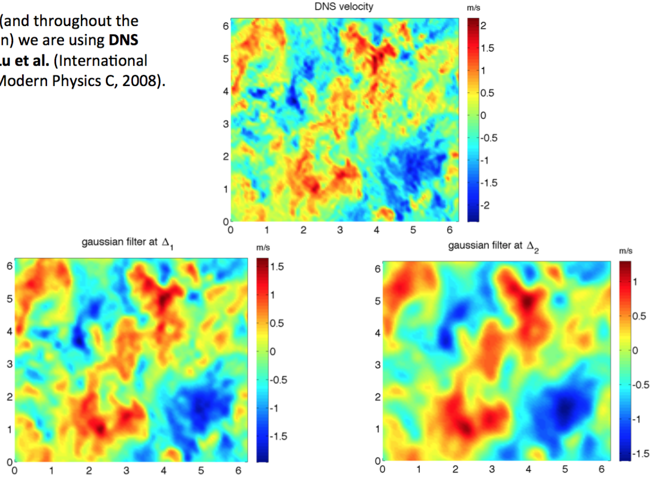

Filtering turbulence (real space, box filter)

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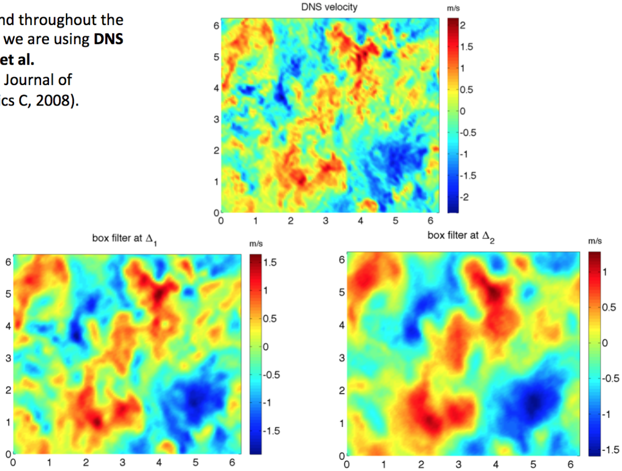
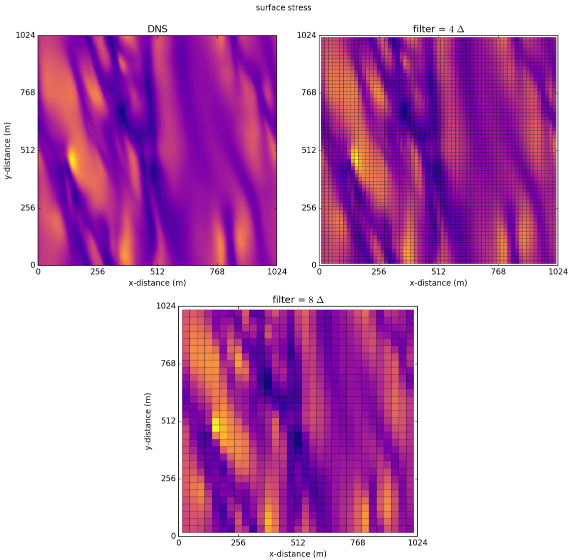


Another example (Fedorovich and Gibbs, submitted)



Filtering turbulence (real space)

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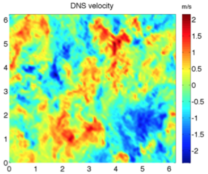
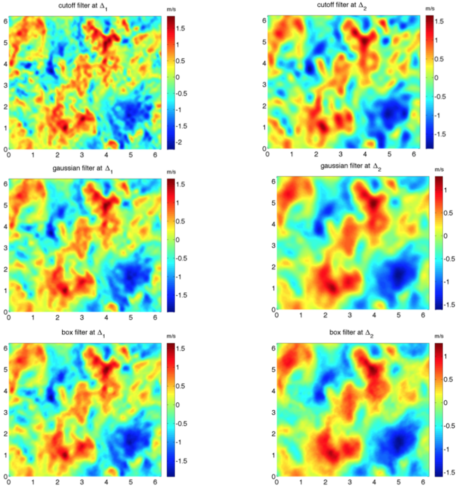
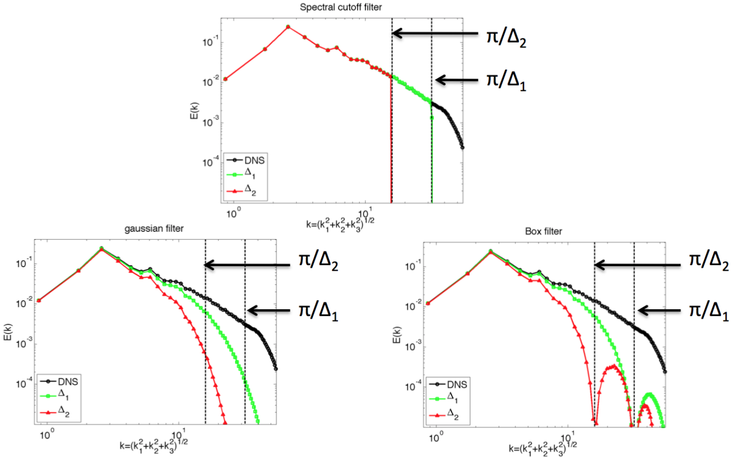
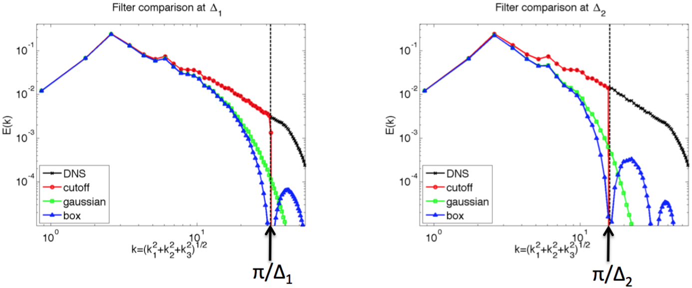




Filtering turbulence (wave space)



Filtering turbulence (wave space)



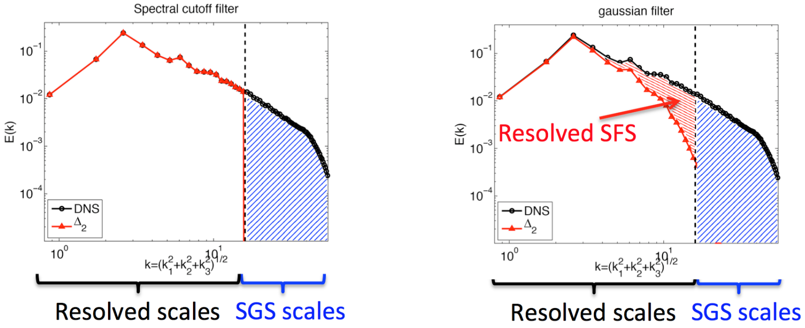
Decomposition of turbulence for real filters

- The LES filter can be used to decompose the velocity field into resolved and subfilter scale (SFS) components:

$$\underbrace{\phi(\vec{x},t)}_{\text{total}} = \underbrace{\tilde{\phi}(\vec{x},t)}_{\text{resolved}} + \underbrace{\phi'(\vec{x},t)}_{\text{subfilter}}$$

- We can use our filtered DNS fields to look at how the choice of our filter kernel affects this separation in wavespace.

Decomposition of turbulence for real filters



- The Gaussian (or box) filter does not have as compact of support in wavespace as the cutoff filter.
- This results in attenuation of energy at scales larger than the filter scale.
- The scales affected by the attenuation are referred to as *resolved SFSs*.