J. Fluid Mech. (1992), *vol.* 238, *pp.* 325–336
Printed in Great Britain

Turbulence: the filtering approach

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(Received 16 July 1990 and in revised form 21 October 1991)

Explicit or implicit filtered representations of chaotic fields like spectral cut-offs or numerical discretizations are commonly used in the study of turbulence and particularly in the so-called large-eddy simulations. Peculiar to these representations is that they are produced by different filtering operators at different levels of resolution, and they can be hierarchically organized in terms of a characteristic parameter like a grid length or a spectral truncation mode. Unfortunately, in the case of a general implicit or explicit filtering operator the Reynolds rules of the mean are no longer valid, and the classical analysis of the turbulence in terms of mean values and fluctuations is not so simple.

In this paper a new operatorial approach to the study of turbulence based on the general algebraic properties of the filtered representations of a turbulence field at different levels is presented. The main results of this analysis are the averaging invariance of the filtered Navier–Stokes equations in terms of the generalized central moments, and an algebraic identity that relates the turbulent stresses at different levels. The statistical approach uses the idea of a decomposition in mean values and fluctuations, and the original turbulent field is seen as the sum of different contributions. On the other hand this operatorial approach is based on the comparison of different representations of the turbulent field at different levels, and, in the opinion of the author, it is particularly fitted to study the similarity between the turbulence at different filtering levels. The best field of application of this approach is the numerical large-eddy simulation of turbulent flows where the large scale of the turbulent field is captured and the residual small scale is modelled. It is natural to define and to extract from the resolved field the resolved turbulence and to use the information that it contains to adapt the subgrid model to the real turbulent field. Following these ideas the application of this approach to the large-eddy simulation of the turbulent flow has been produced (Germano *et al.* 1991). It consists in a dynamic subgrid-scale eddy viscosity model that samples the resolved scale and uses this information to adjust locally the Smagorinsky constant to the local turbulence.

1. Introduction

A filtered representation of an original chaotic field u_i can be formally and quite generally expressed by a convolutional integral (Leonard 1974) given by

$$\langle u_i(\boldsymbol{x},t)\rangle_{l,\theta} = \int u_i(\boldsymbol{x}',t')\,\mathscr{G}(\boldsymbol{x}-\boldsymbol{x}',t-t';l,\theta)\,\mathrm{d}^3\boldsymbol{x}'\,\mathrm{d}t', \tag{1}$$

with

$$\int \mathscr{G}(\boldsymbol{x}-\boldsymbol{x}',t-t';l,\theta)\,\mathrm{d}^3\boldsymbol{x}'\,\mathrm{d}t' = 1, \tag{2}$$

where l and θ are a characteristic filter length and a characteristic filter time. Typical implicit filters are the numerical discretizations, and characteristic explicit filters are the following:

$$\langle u_i(\boldsymbol{x},t)\rangle_\theta = \frac{1}{\theta}\int_t^{t+\theta} u_i(\boldsymbol{x},t')\,\mathrm{d}t', \tag{3}$$

$$\langle u_i(\boldsymbol{x},t)\rangle_l = \frac{1}{l^3}\int_x^{x+l}\int_y^{y+l}\int_z^{z+l} u_i(\boldsymbol{x}',t)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z', \tag{4}$$

while typical filters commonly used in the large-eddy simulation of turbulent flows are the spectral cutoffs and the Gaussian filters given by the convolutional nucleus

$$\mathscr{G}(\boldsymbol{x}-\boldsymbol{x}';l) = \left(\frac{6}{\pi l^2}\right)^{\frac{3}{2}} \exp\left(-\frac{6(\boldsymbol{x}-\boldsymbol{x}')^2}{l^2}\right). \tag{5}$$

Formally, the filtering approach stands between the direct approach and the statistical approach and probably will produce in the future a unified theory linking the direct approach to the statistical one by a continuous interval of intermediate steps. We notice that physical space–time averages are often substitutive of the ensemble average, particularly when symmetries or homogeneities are present in the flow. As a particular example let us consider the class of time filters (3) parameterized in terms of the characteristic filter time θ; we see that they constitute a hierarchy of filters going from the identity, $\theta = 0$, the direct approach, to the infinite time average, $\theta \rightarrow \infty$, and it is well known that for statistically steady flows this time average converges to the ensemble average.

From an operatorial point of view the filtering approach consists in applying explicitly or implicitly to the Navier–Stokes equations a linear operator commutative with the space and time derivatives. Its principal field of application is in computational fluid dynamics, and the characteristic filter lengths and times are intimately related to the grid discretization or to the spectral truncation. However, the history of the filtering approach is very old, and dates from the first studies on turbulence. The first average proposed by Boussinesq (1877) is given by (3), where θ is 'un temps assez petit', while Reynolds (1895) preferred the spatial average (4), where l^3 is a certain volume of space. It is evident that only in the case $\theta \rightarrow \infty$ or $l \rightarrow \infty$ do these averaging operators satisfy the simple conditions that Reynolds himself stated as necessary for a well-behaved mean operator:

$$\langle f\langle g\rangle\rangle = \langle f\rangle\langle g\rangle, \tag{6}$$

$$\langle\langle f\rangle\rangle = \langle f\rangle, \tag{7}$$

and so attention was subsequently directed to the statistical averages that clearly satisfy the Reynolds rules of the mean (6), (7). In the shadow of the statistical approach the filtering approach received little attention: papers are very scarce and good reviews of what was done up to the advent of the computer can be found in Kampè de Fèriet (1957) and Monin & Yaglom (1971).

This attitude radically changed with the advent of the computer. The analogies between the filtering operators and the numerical discretization were appreciated (Rogallo & Moin 1984); it was shown that their characteristic lengths and times can be correlated with computational grid values and the filtering approach became the framework that permitted a formal theory of the large-eddy simulations in all its aspects. Obviously the difficulties regarding averages that do not satisfy the Reynolds rules of the mean remained. The old idea of averaging the Navier–Stokes equations was almost always coupled to the parallel idea of a decomposition of the turbulent signal into a mean part $\langle u_i \rangle$, that is to say the part generated by the average, and a fluctuation u_i', whose sum is the original quantity, $u_i = \langle u_i \rangle + u_i'$. The usual procedure of the statistical approach is to write equations for the fluctuating velocities u_i' and to produce evolutionary equations for the central moments defined in terms of the fluctuations as

$$\langle u_i' u_j' \rangle, \langle u_i' u_j' u_k' \rangle, \ldots, \tag{8}$$

giving rise to the well-known problem of closure. This procedure, when extended to non-Reynolds averaging operators (Germano 1987), produces a lot of problems mainly because the mean value of the fluctuations is now different from zero and the assumption that there is no correlation between the mean values and the fluctuations is no longer valid:

$$\langle u_i' \rangle \neq 0, \quad \langle\langle u_i \rangle u_j' \rangle \neq 0. \tag{9}$$

As a consequence the classical relations between the moments $\langle u_i u_j \rangle, \langle u_i u_j u_k \rangle, \ldots$ and the central moments (Monin & Yaglom 1971, p. 223)

$$\left.\begin{aligned} \langle u_i' u_j' \rangle &= \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle, \\ \langle u_i' u_j' u_k' \rangle &= \langle u_i u_j u_k \rangle - \langle u_i \rangle \langle u_j' u_k' \rangle - \langle u_j \rangle \langle u_k' u_i' \rangle - \langle u_k \rangle \langle u_i' u_j' \rangle \\ &\quad - \langle u_i \rangle \langle u_j \rangle \langle u_k \rangle, \\ \langle u_i' u_j' u_k' u_l' \rangle &= \ldots \end{aligned}\right\} \tag{10}$$

are no longer valid, and new terms arise that considerably complicate the system of averaged equations and their closure. However, if we introduce a new set of generalized central moments

$$\tau(u_j, u_k), \tau(u_i, u_j, u_k), \ldots \tag{11}$$

formally defined by (10):

$$\left.\begin{aligned} \tau(u_i, u_j) &= \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle, \\ \tau(u_i, u_j, u_k) &= \langle u_i u_j u_k \rangle - \langle u_i \rangle \tau(u_j, u_k) \\ &\quad - \langle u_j \rangle \tau(u_k, u_i) - \langle u_k \rangle \tau(u_i, u_j) - \langle u_i \rangle \langle u_j \rangle \langle u_k \rangle, \\ \tau(u_i, u_j, u_k, u_l) &= \ldots, \end{aligned}\right\} \tag{12}$$

simplicity is regained, as we will see. A trace of this idea can be found in the papers of Lilly (1966) and Deardorff (1970). In this last paper, when the Reynolds rules of the mean are assumed valid in the case of the box filter (4), he states: 'However, this assumption is not separately necessary and may be incorporated into later assumptions if $\langle u_i' u_j' \rangle$ is formally replaced by $\langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle$ wherever it appears.' In the next section we will apply this formal replacement in a rigorous way, and the evolutionary equations for the generalized central moments will be deduced for a general linear filtering operator. We anticipate that the result will be very simple and surprising at the same time: the evolutionary equations of the generalized central moments are exactly the Reynolds equations, and the algebraic structure of the closure is the same for every linear commuting filter. We call this the *averaging invariance* of the turbulent equations.

2. The averaging invariance of the turbulent equations in terms of the generalized central moments

Let us consider a generic linear and constant-preserving averaging operator

$$\langle f+g\rangle = \langle f\rangle + \langle g\rangle, \tag{13}$$

$$\langle \alpha f\rangle = \alpha\langle f\rangle \quad \text{if } \alpha = \text{constant} \tag{14}$$

having only the commuting properties with space and time derivatives

$$\langle f_{,t}\rangle = \langle f\rangle_{,t}; \quad \langle f_{,k}\rangle = \langle f\rangle_{,k}. \tag{15}$$

If we now consider the Navier–Stokes equations for incompressible fluids

$$u_{k,k} = 0, \tag{16}$$

$$u_{i,t} + (u_i u_k)_{,k} = -p_{,i} + \sigma_{ik,k}, \tag{17}$$

and the equations derived by taking a moment of (17) with u_j and adding this to another moment of the same equation but with the indices interchanged,

$$(u_i u_j)_{,t} + (u_i u_j u_k)_{,k} = -[p u_i \delta_{jk} + p u_j \delta_{ik} - \nu(u_i u_j)_{,k}]_{,k} + 2p s_{ij} - 2\nu u_{i,k} u_{j,k}, \tag{18}$$

where $$\sigma_{ij} = 2\nu s_{ij}; \quad s_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{19}$$

it is, first, very easy to see that in terms of the moments $\langle u_i u_j\rangle, \langle u_i u_j u_k\rangle, \ldots$ the filtered equations are the same for every filter. What is, however, more interesting is the fact that this averaging invariance can be extended directly and without recourse to the fluctuations to the generalized central moments that are the usual quantities modelled in the closure problem. By some simple algebra we can recover the following equations:

$$\langle u_k\rangle_{,k} = 0, \tag{20}$$

$$\langle u_i\rangle_{,t} + (\langle u_i\rangle\langle u_k\rangle)_{,k} = -\langle p\rangle_{,i} + \langle \sigma_{ik}\rangle_{,k} - [\tau(u_i, u_k)]_{,k}, \tag{21}$$

$$\begin{aligned}
[\tau(u_i, u_j)]_{,t} + [\tau(u_i, u_j)\langle u_k\rangle]_{,k} = &-\{\tau(u_i, u_j, u_k) \\
&+ \tau(p, u_i)\delta_{jk} + \tau(p, u_j)\delta_{ik} - \nu[\tau(u_i, u_j)]_{,k}\}_{,k} \\
&+ 2\tau(p, s_{ij}) - 2\nu\tau(u_{i,k}, u_{j,k}) \\
&- \tau(u_i, u_k)\langle u_j\rangle_{,k} - \tau(u_j, u_k)\langle u_i\rangle_{,k},
\end{aligned} \tag{22}$$

$$[\tau(u_i, u_j, u_k)]_{,t} + [\tau(u_i, u_j, u_k)\langle u_l\rangle]_{,l} = \ldots, \tag{23}$$

where the generalized central moments $\tau(f,g), \tau(f,g,h), \ldots$ are defined as in (12):

$$\begin{aligned}
\tau(f,g) &= \langle fg\rangle - \langle f\rangle\langle g\rangle, \\
\tau(f,g,h) &= \langle fgh\rangle - \langle f\rangle\tau(g,h) - \langle g\rangle\tau(h,f) - \langle h\rangle\tau(f,g) - \langle f\rangle\langle g\rangle\langle h\rangle, \\
\tau(f,g,h,k) &= \ldots,
\end{aligned} \tag{24}$$

and we notice that the contracted form of (22) gives a generalized equation for the turbulent energy:

$$E_{\mathrm{T},t} + (E_{\mathrm{T}}\langle u_k\rangle)_{,k} = -[\tfrac{1}{2}\tau(u_i, u_i, u_k) + \tau(p, u_k) - \nu E_{T,k}]_{,k} - \nu\tau(u_{i,k}, u_{i,k}) - \tau(u_i, u_k)\langle s_{ik}\rangle, \tag{25}$$

where E_{T} is the generalized turbulent energy given by

$$E_{\mathrm{T}} = \tfrac{1}{2}\tau(u_i, u_i). \tag{26}$$

It is easy to see that the structure of the averaged equations in terms of the generalized central moments does not depend on the particular filter and is formally equal to the structure of the well-known statistical equations (Davydov 1961). Apart from the particular form that the generalized turbulent stresses assume in the case of a statistical averaging operator subjected to the Reynolds rules of the mean, we notice that the structure of the filtered equations is invariant to the particular averaging operation. We will refer to this property of the filtered equations as their *averaging invariance*, and we notice again that this invariance is in terms of the generalized central moments previously defined in (24). They generalize the usual statistical central moments expressed in terms of the fluctuations and given by the relations

$$\tau(f,g) = \langle f'g' \rangle, \quad \tau(f,g,h) = \langle f'g'h' \rangle. \tag{27}$$

3. An algebraic property of the generalized central moments. The resolved turbulence

The averaging invariance of the turbulent equations is, in the opinion of the author, largely unexplored and in a sense more suffered than tested and appreciated. We will not discuss now the implications that the averaging invariance has on the closure problem, and only note that in some way or another it suggests a large indifference of the equations to the implicit or explicit filter actually applied in a single-level filtered representation.

If the one-level filtered equations are independent of what real filter is applied, we can explore multi-level filtering procedures in order to generate improved subgrid models. Usually the multi-level procedures are based on spectral splitting operators, and we refer to the papers of Tchen (1973) and Schiestel (1987) on the matter. We notice that multi-level procedures have usually been produced in terms of a multiple decomposition of the velocity field u_i in ranks or in components $u_i^{(\alpha)}$ that when summed reproduce the original field

$$u_i = \sum_\alpha u_i^{(\alpha)}. \tag{28}$$

In this paper we prefer to compare what happens at different levels, and there will be no recourse to any kind of decomposition. In a sense this approach arises naturally when we do numerical computations: we test what happens if we change our grid intervals, and compare the results at different levels of resolution. The tool that will help us in this approach is an algebraic relation that we now will deduce and illustrate.

As we have seen, the main problem of a large-eddy simulation is to model the generalized turbulent stress $\tau_f(u_i, u_j)$ related to the two velocity components u_i, u_j and defined

$$\tau_f(u_i, u_j) = \langle u_i u_j \rangle_f - \langle u_i \rangle_f \langle u_j \rangle_f, \tag{29}$$

where F is now the particular implicit or explicit filter applied and $\langle u_i \rangle_f$ the F-level filtered values. Let us now introduce another filter, the explicit test filter G, and let us denote by $\langle u_i \rangle_{fg}$,

$$\langle u_i \rangle_{fg} = \langle\langle u_i \rangle_f \rangle_g = \langle\langle u_i \rangle_g \rangle_f, \tag{30}$$

the $FG = GF$ filtered values. If we denote by $\tau_{fg}(u_i, u_j)$ the turbulent stress at the FG-level,

$$\tau_{fg}(u_i, u_j) = \langle u_i u_j \rangle_{fg} - \langle u_i \rangle_{fg} \langle u_j \rangle_{fg}, \tag{31}$$

and by $\tau_g(\langle u_i\rangle_f, \langle u_j\rangle_f)$ the resolved turbulent stress extracted from the resolved scale F,

$$\tau_g(\langle u_i\rangle_f, \langle u_j\rangle_f) = \langle\langle u_i\rangle_f \langle u_j\rangle_f\rangle_g - \langle u_i\rangle_{fg} \langle u_j\rangle_{fg}, \tag{32}$$

the following algebraic relation holds:

$$\tau_{fg}(u_i, u_j) = \langle \tau_f(u_i, u_j)\rangle_g + \tau_g(\langle u_i\rangle_f, \langle u_j\rangle_f). \tag{33}$$

The physical meaning of this algebraic relation is simple: the turbulent stress at the FG-level is equal to the G-averaged value of the turbulent stress at the F-level plus the resolved turbulent stress $\tau_g(\langle u_i\rangle_f, \langle u_j\rangle_f)$ extracted from the resolved scale F. Similarly we can extract from the resolved scale the resolved turbulent energy, or the resolved production, or the resolved dissipation, or anything that we would like to test. We notice that the algebraic relation (33) applies locally in space and time, so that the resolved turbulence is composed of fluctuating terms. In the applications we will see the utility of this algebraic relation in an ensemble form. If we denote an ensemble average with an overline, as usual, we can also write

$$\overline{\tau_{fg}(u_i, u_j)} = \overline{\langle \tau_f(u_i, u_j)\rangle_g} + \overline{\tau_g(\langle u_i\rangle_f, \langle u_j\rangle_f)}. \tag{34}$$

It is also interesting to apply (33) to the case in which the test filter G is the ensemble average E. We obtain

$$\tau_{fe}(u_i, u_j) = \overline{\tau_f(u_i, u_j)} + \tau_e(\langle u_i\rangle_f, \langle u_j\rangle_f), \tag{35}$$

and in the particular case in which $EF = E$ we have

$$\tau_e(u_i, u_j) = \overline{\tau_f(u_i, u_j)} + \tau_e(\langle u_i\rangle_f, \langle u_j\rangle_f). \tag{36}$$

We notice that $\tau_e(u_i, u_j)$ represents the usual Reynolds stress,

$$\tau_e(u_i, u_j) = \overline{u_i u_j} - \bar{u}_i \bar{u}_j, \tag{37}$$

so that (36) can be interpreted as follows: the Reynolds stress is equal to the ensemble value of the turbulent stress at the F-level plus the resolved turbulent stress $\tau_e(\langle u_i\rangle_f, \langle u_j\rangle_f)$. We notice that the resolved turbulent stresses can be explicitly calculated in a large-eddy simulation and in the following the possible use of this algebraic property in multi-level subgrid modelling is discussed.

4. A dynamic procedure for the determination of the Smagorinsky constant

Explicit spectral cutoff filters and Gaussian filters (Leonard 1974), volume averages (Schumann 1975), and statistical filters (Yoshizawa 1989) have been proposed and used in the past and belong to the category of filters considered in this paper. Also, numerical discretization can be interpreted as an implicit filter having generally unknown properties (Rogallo & Moin 1984), and explicit prefiltering has been suggested in order to remove the indeterminacy of the numerical discretization. Apart from the exact determination of the real filtering operator represented by the particular large-eddy-simulation technique adopted, we think that the problem of the closure and the related subgrid models for the turbulent stresses can be largely discussed in the framework of the operatorial approach previously described.

A basic ingredient that characterizes the subgrid models used in the large-eddy-simulation is that their characteristic lengthscale is usually given by the grid size. As a consequence simple algebraic models or at most one-equation models for the turbulent energy have been adopted. Among them the simplest one is the Smagorinsky (1963) model whose basic ingredients are the following. We assume that the anisotropic, deviatoric part of the turbulent stress $\tau_f^a(u_i, u_j)$,

$$\tau_f^a(u_i, u_j) = \tau_f(u_i, u_j) - \tfrac{1}{3}\delta_{ij}\tau_f(u_l, u_l), \tag{38}$$

is related to the averaged strain-rate tensor $\langle s_{ij}\rangle_f$,

$$\langle s_{ij}\rangle_f = \tfrac{1}{2}(\langle u_i\rangle_{f,j} + \langle u_j\rangle_{f,i}) \tag{39}$$

by the constitutive relation

$$\tau_f^a(u_i, u_j) = -2\nu_f \langle s_{ij}\rangle_f, \tag{40}$$

where ν_f is the eddy viscosity related to the generic explicit filtering operator F that produces the filtered values $\langle u_j\rangle_f$. From the dimensional analysis, this eddy viscosity can be given in terms of the characteristic length of the filter l_f and the dissipation ϵ by the relation

$$\nu_f = c l_f^{\frac{4}{3}} \epsilon^{\frac{1}{3}}, \tag{41}$$

and we finally assume that the turbulent dissipation ϵ is in equilibrium with the turbulent production P. From the transport equation of the generalized turbulent energy (25), the production term P is given by

$$P = -\tau_f(u_l, u_m)\langle s_{lm}\rangle_f = -\tau_f^a(u_l, u_m)\langle s_{lm}\rangle_f, \tag{42}$$

and if we write that $\epsilon = P$ we obtain

$$\nu_f = c_S l_f^2 (2\langle s_{lm}\rangle_f \langle s_{lm}\rangle_f)^{\frac{1}{2}}, \tag{43}$$

where c_S is the Smagorinsky constant. Note that the basic ingredients of this single-level model are averaging invariant. It is plausible to suppose that the structure of the Smagorinsky model is largely independent of the particular average used, having only its commutivity with the derivatives, and that the particular average explicitly appears in the scale of the model, that is to say the values of c_S and l_f. Obviously in this way the consistency between the filter and the model is assured, mainly by the fact that the strain rate tensor that appears in it is consistently averaged by the same operator. This point also applies when no explicit filtering operators are used if we assume that a particular numerical scheme is equivalent to a linear truncating or filtering operator commuting with the derivatives.

From the beginning (Lilly 1966; Deardorff 1970), the eddy viscosity model has been intensively used in large-eddy simulations of turbulent flows owing to its balanced mixture of physical content and mathematical simplicity. Its extension from homogeneous and isotropic turbulence to homogeneous turbulence in sheared and rotating flows has, however, created some problems, and more will probably arise in its application to transitional flows and compressible turbulence. Evidence exists (Rogallo & Moin 1984) that the Smagorinsky constant decreases in the presence of mean shear, where the large-scale mean velocity gradient is probably overestimated, and it must be supplemented at the wall by empirical wall functions. Generally speaking we can say that this model cannot recognize if the flow is laminar at the actual computational level, and these are its limits in a pure single-level representation.

In order to extend the range of application of the Smagorinsky model we can improve the consistency of the model with the results that are obtained and adjust the model to them in an interactive way. There is obviously in a large-eddy simulation an enormous amount of information that could be selectively used in order to adapt the model to the real particular flow considered. For simplicity we assume that F, G and the product $GF = FG$ are space-filtering or truncating operators characterized in the physical space by their shape and by their characteristic lengths l_f, l_g and l_{fg}. One of the problems of subgrid modelling consists in this case in determining the appropriate values of the constant c_S that appears in the Smagorinsky model. We recall that an attractive property of the explicit Gaussian filters is that the commutative product $FG = GF$ of two Gaussian filters F, G with characteristic lengths l_f and l_g is another Gaussian filter with a characteristic length l_{fg} given by

$$l_{fg} = (l_f^2 + l_g^2)^{\frac{1}{2}}, \tag{44}$$

and the same happens in the case of the product of two cutoff spectral filters characterized in physical space by their characteristic lengths l_f and l_g. In this latter case their product is another spectral cutoff filter with a characteristic length l_{fg} given by

$$\langle\langle u_i \rangle_f \rangle_g = \langle\langle u_i \rangle_g \rangle_f = \langle u_i \rangle_f; \quad l_{fg} = l_f \quad \text{if} \quad l_f > l_g, \tag{45}$$

$$\langle\langle u_i \rangle_f \rangle_g = \langle\langle u_i \rangle_g \rangle_f = \langle u_i \rangle_g; \quad l_{fg} = l_g \quad \text{if} \quad l_f < l_g. \tag{46}$$

In order to apply the previous ideas concerning a possible multi-level dynamic procedure to the Smagorinsky model, let us substitute into the algebraic identity (33) the values given by the Smagorinsky model. We can write

$$\tau_f^{\mathrm{a}}(u_i, u_j) = -2\nu_f \langle s_{ij} \rangle_f, \tag{47}$$

$$\nu_f = c_S\, l_f^2 (2\langle s_{lm} \rangle_f \langle s_{lm} \rangle_f)^{\frac{1}{2}}, \tag{48}$$

$$\tau_{fg}^{\mathrm{a}}(u_i, u_j) = -2\nu_{fg} \langle s_{ij} \rangle_{fg}, \tag{49}$$

$$\nu_{fg} = c_S\, l_{fg}^2 (2\langle s_{lm} \rangle_{fg} \langle s_{lm} \rangle_{fg})^{\frac{1}{2}}, \tag{50}$$

and we finally obtain the relation

$$-2\nu_{fg} \langle s_{ij} \rangle_{fg} = -2\langle \nu_f \langle s_{ij} \rangle_f \rangle_g + \tau_g^{\mathrm{a}}(\langle u_i \rangle_f, \langle u_j \rangle_f), \tag{51}$$

where $\tau_g^{\mathrm{a}}(\langle u_i \rangle_f, \langle u_j \rangle_f)$ is the anisotropic, deviatoric part of the resolved turbulent stress

$$\tau_g^{\mathrm{a}}(\langle u_i \rangle_f, \langle u_j \rangle_f) = \tau_g(\langle u_i \rangle_f, \langle u_j \rangle_f) - \tfrac{1}{3}\delta_{ij}\, \tau_g(\langle u_l \rangle_f, \langle u_l \rangle_f), \tag{52}$$

which can be used in order to determine the local Smagorinsky constant. We notice that the 'constant' so calculated depends on the position, the time and the indices i,j, so that it is not a constant at all. As regards the dependence on the indices i,j a scalar procedure should probably only pretend that the production-dissipation P_{fg} at the FG level,

$$P_{fg} = -\tau_{fg}(u_l, u_m) \langle s_{lm} \rangle_{fg} = -\tau_{fg}^{\mathrm{a}}(u_l, u_m) \langle s_{lm} \rangle_{fg}, \tag{53}$$

is consistent with the sample. In this case we can write the contracted relation

$$-2\nu_{fg} \langle s_{lm} \rangle_{fg} \langle s_{lm} \rangle_{fg} = -2\langle \nu_f \langle s_{lm} \rangle_f \rangle_g \langle s_{lm} \rangle_{fg} + \tau_g^{\mathrm{a}}(\langle u_l \rangle_f, \langle u_m \rangle_f) \langle s_{lm} \rangle_{fg}, \tag{54}$$

and a local isotropic value of the Smagorinsky constant can be obtained. This particular model was proposed by the author (Germano 1990) at the CTR 1990 Summer Meeting, and the interaction theory computation (Germano *et al.* 1991) has greatly improved this suggested procedure as follows. It was clear from the first numerical results that this dynamic Smagorinsky constant should be obtained in an ensemble form. Such a form can be easily obtained by a time average for a statistically steady flow or a space average if some symmetry plane for the turbulence exists. If we indicate that ensemble average by an overline we can write the expression (54) in the form

$$-2\overline{\nu_{fg}\langle s_{lm}\rangle_{fg}\langle s_{lm}\rangle_{fg}} = -2\overline{\langle \nu_f\langle s_{lm}\rangle_f\rangle_g\langle s_{lm}\rangle_{fg}} + \overline{\tau_g^{a}(\langle u_l\rangle_f, \langle u_m\rangle_f)\langle s_{lm}\rangle_{fg}}. \quad (55)$$

The Smagorinsky constant c_S most consistent with the model can be easily derived from this equation as

$$c_S = \frac{\overline{\tau_g^{a}(\langle u_l\rangle_f, \langle u_m\rangle_f)\langle s_{lm}\rangle_{fg}}}{2l_f^2\overline{\langle s_f\langle s_{lm}\rangle_f\rangle_g\langle s_{lm}\rangle_{fg}} - 2l_{fg}^2\overline{s_{fg}\langle s_{lm}\rangle_{fg}\langle s_{lm}\rangle_{fg}}}, \quad (56)$$

and we can finally write explicitly

$$\tau_f^{a}(u_i, u_j) = \frac{\overline{\tau_g^{a}(\langle u_l\rangle_f, \langle u_m\rangle_f)\langle s_{lm}\rangle_{fg}}}{\dfrac{l_{fg}^2}{l_f^2}\overline{s_{fg}\langle s_{lm}\rangle_{fg}\langle s_{lm}\rangle_{fg}} - \overline{\langle s_f\langle s_{lm}\rangle_f\rangle_g\langle s_{lm}\rangle_{fg}}}\, s_f\langle s_{ij}\rangle_f, \quad (57)$$

where s_f and s_{fg} are given by

$$s_f = (2\langle s_{lm}\rangle_f\langle s_{lm}\rangle_f)^{\frac{1}{2}}, \quad (58)$$

$$s_{fg} = (2\langle s_{lm}\rangle_{fg}\langle s_{lm}\rangle_{fg})^{\frac{1}{2}}. \quad (59)$$

Note that the model given by (57) depends only on the ratio of the filter lengths l_{fg}/l_f, and it goes to zero with the resolved turbulent stress $\tau_g^{a}(\langle u_i\rangle_f, \langle u_j\rangle_f)$, so that it is able to recognize when the flow is laminar. The optimal size of the test filter G should be chosen carefully. If its length l_g is small with respect to the filtering length l_f we have to expect some form of ill-conditioning; on the other hand, if this length is large the test is probably biased by the increasing predominance of the large resolved scales. This dynamic subgrid-scale eddy viscosity model was implemented by applying explicit spectral cutoff filters and it was tested *a priori* by using the direct numerical summation database of Kim, Moin & Moser (1987) for turbulent channel flow and that of Zang, Gilbert & Kleiser (1990) for transitional flow. In order to determine its accuracy it was also tested *a posteriori* in the large-eddy simulation of transitional and fully developed turbulent channel flow. The results are in satisfactory agreement with the direct simulation data.

5. Consistent decomposition of the generalized central moments

We have seen that the structure of the averaged equations in terms of the generalized central moments does not depend on the particular filter and that they are homomorphic to the statistically averaged Reynolds equations. We have also noted that the generalized central moments of the second order, $\tau(u_i, u_j)$, reduce in the case of a statistical operator to the so-called turbulent or Reynolds stresses and we have discussed the relations between different turbulent stresses at different resolution levels. Let us now discuss other interesting algebraic properties of these quantities. First, it is very easy to see that the generalized central moments $\tau(f,g)$, $\tau(f,g,h), \ldots$ are symmetric in their arguments, with some of the following algebraic properties:

$$\tau(f,\alpha) = 0 \quad \text{if} \quad \alpha = \text{constant}, \quad (60)$$

$$\tau(f,g,\alpha) = 0 \quad \text{if} \quad \alpha = \text{constant}, \quad (61)$$

$$\tau(f,g)_{,t} = \tau(f_{,t}, g) + \tau(f, g_{,t}), \quad (62)$$

$$\tau(f,g)_{,k} = \tau(f_{,k}, g) + \tau(f, g_{,k}). \quad (63)$$

If we now decompose f, g, h as $f = f_1 + f_2, g = g_1 + g_2, h = h_1 + h_2$, the related homogeneous and consistent (in the sense defined below (69)) decomposition of the generalized central moments is

$$\tau(f_1+f_2, g_1+g_2) = \tau(f_1, g_1) + \tau(f_1, g_2) + \tau(f_2, g_1) + \tau(f_2, g_2), \tag{64}$$

$$\begin{aligned}\tau(f_1+f_2, g_1+g_2, h_1+h_2) &= \tau(f_1, g_1, h_1) + \tau(f_1, g_1, h_2) + \tau(f_1, g_2, h_1) \\ &\quad + \tau(f_2, g_1, h_1) + \tau(f_2, g_1, h_2) + \tau(f_2, g_2, h_1) + \tau(f_1, g_2, h_2) + \tau(f_2, g_2, h_2),\end{aligned} \tag{65}$$

and as a consequence we recover the Galilean invariance of the quantities $\tau(f, g)$, $\tau(f, g, h), \ldots$. We have

$$\left.\begin{aligned}\tau(f+\alpha, g+\beta) &= \tau(f, g), \\ \tau(f+\alpha, g+\beta, h+\gamma) &= \tau(f, g, h),\end{aligned}\right\} \tag{66}$$

where α, β, γ are constants, and in particular if we now decompose each quantity f, g, h into a mean value $\langle f\rangle, \langle g\rangle, \langle h\rangle$ and a fluctuation f', g', h',

$$f = \langle f\rangle + f'; \quad g = \langle g\rangle + g'; \quad h = \langle h\rangle + h', \tag{67}$$

we can consistently decompose the quantities $\tau(f, g)$ and $\tau(f, g, h)$ as follows:

$$\tau(\langle f\rangle + f', \langle g\rangle + g') = \tau(\langle f\rangle, \langle g\rangle) + \tau(\langle f\rangle, g') + \tau(f', \langle g\rangle) + \tau(f', g'), \tag{68}$$

$$\begin{aligned}\tau(\langle f\rangle + f', \langle g\rangle + g', \langle h\rangle + h') &= \tau(\langle f\rangle, \langle g\rangle, \langle h\rangle) + \tau(\langle f\rangle, \langle g\rangle, h') + \tau(\langle f\rangle, g', \langle h\rangle) \\ &\quad + \tau(f', \langle g\rangle, \langle h\rangle) + \tau(\langle f\rangle, g', h') + \tau(f', \langle g\rangle, h') + \tau(f', g', \langle h\rangle) + \tau(f', g', h').\end{aligned} \tag{69}$$

Note that this decomposition of the generalized turbulent stresses is consistent with their definition. All terms have the same properties, and particularly they are Galilean invariant term by term. We call this decomposition *consistent decomposition*, and we refer to the terms $\tau(\langle f\rangle, \langle g\rangle)$ as *resolved turbulent stresses*, to the terms $\tau(f', g')$ as *subgrid stresses*, while the remaining terms are the *cross-stresses*, since in a large-eddy simulation the mean values are related to the values resolved at grid level. Until now we have not used the concept of a decomposition of the turbulent quantities into two parts, mean and fluctuating and we have preferred to show the algebraic properties that relate and compare the turbulent stresses at different levels of resolution. As a consequence it is useful here to recall that if we decompose the turbulent quantities as

$$u_i = \langle u_i\rangle_f + u_i'; \quad u_j = \langle u_j\rangle_f + u_j', \tag{70}$$

the classical decomposition is given by (Leonard 1974)

$$\tau_f(u_i, u_j) = L_{ij} + C_{ij} + R_{ij}, \tag{71}$$

where

$$L_{ij} = \langle\langle u_i\rangle_f \langle u_j\rangle_f\rangle_f - \langle u_i\rangle_f \langle u_j\rangle_f \tag{72}$$

is the resolved or Leonard term,

$$C_{ij} = \langle u_i' \langle u_j\rangle_f\rangle_f + \langle u_j' \langle u_i\rangle_f\rangle_f \tag{73}$$

the cross-term, and

$$R_{ij} = \langle u_i' u_j'\rangle_f \tag{74}$$

the Reynolds term. We notice that following the arguments developed in this paper these terms are not turbulent stresses and, as we have seen, a decomposition consistent with the averaging invariance of the turbulent equation is given by

$$\tau_f(u_i, u_j) = \mathscr{L}_{ij} + \mathscr{C}_{ij} + \mathscr{R}_{ij}, \tag{75}$$

where now

$$\mathscr{L}_{ij} = \tau_f(\langle u_i\rangle_f, \langle u_j\rangle_f) \tag{76}$$

is a resolved turbulent stress, and

$$\left.\begin{aligned}\mathscr{C}_{ij} &= \tau_f(u_i', \langle u_j\rangle_f)+\tau_f(\langle u_i\rangle_f, u_j'),\\ \mathscr{R}_{ij} &= \tau_f(u_i', u_j').\end{aligned}\right\} \tag{77}$$

We remark again that an important difference between the two decomposition is that the first is composed of terms that are not singularly Galilean invariant (Speziale 1985), while in the second one the terms obviously have the same properties as discussed in the previous sections so that they preserve one by one the Galilean invariance of the decomposed original stress (Germano 1986).

6. Conclusions

In this paper some algebraic properties of filtering operators have been analysed. It is shown that the averaged equations are the same in terms of the generalized central moments, and the resolved turbulence is defined. Algebraic consistency rules that relate these resolved quantities to the turbulent stresses at different levels are derived, and their possible use in subgrid modelling is examined. The idea of a comparison between different levels complementary to the usual idea of a decomposition in ranks is introduced and discussed.

In the opinion of the author the algebraic relation (33) should be interpreted as a general condition that, in some way or another, a multi-level filtering procedure must satisfy and from the same perspective different multi-level filtering techniques could be suggested for different subgrid models. We finally remark that another attractive application of the multi-level filtering approach concerns its possible use as an experimental tool of investigation. Given a hierarchy of filters organized in terms of a space or time parameter, it could be interesting to analyse and compare the resolved turbulence at different filtering levels. If for example we consider the set of time-box filters given by

$$\langle u_i(\boldsymbol{x},t)\rangle_\theta = \frac{1}{\theta}\int_t^{t+\theta} u_i(\boldsymbol{x},t')\,\mathrm{d}t' \tag{78}$$

and organized in terms of the characteristic time θ, we could conceive a multi-level filtering analysis of the turbulence in terms of the generalized central moments and their properties. In analogy with the wavelet analysis that gives us a local spectral picture of the flow, this *boxlet* approach should provide, at least for statistically steady fields, a local statistical analysis of the turbulence.

The author has been actively involved in this research for some years, and it is a particular pleasure to acknowledge here the constant, confident and lively support and cooperation of the Center for Turbulence Research, Stanford University/Nasa Ames. The financial support from the Ministero Italiano dell'Università e della Ricerca Scientifica e Tecnologica is greatly appreciated.

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