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Prandtl and the Göttingen school

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2.1 Introduction

In the early decades of the 20th century Göttingen was the center for mathematics. The foundations were laid by Carl Friedrich Gauss (1777–1855) who from 1808 was head of the observatory and professor for astronomy at the Georg August University (founded in 1737). At the turn of the 20th century, the well-known mathematician Felix Klein (1849–1925), who joined the University in 1886, established a research center and brought leading scientists to Göttingen. In 1895 David Hilbert (1862–1943) became Chair of Mathematics and in 1902 Hermann Minkowski (1864–1909) joined the mathematics department. At that time, pure and applied mathematics pursued diverging paths, and mathematicians at Technical Universities were met with distrust from their engineering colleagues with regard to their ability to satisfy their practical needs (Hensel, 1989). Klein was particularly eager to demonstrate the power of mathematics in applied fields (Prandtl, 1926b; Manegold, 1970). In 1905 he established an Institute for Applied Mathematics and Mechanics in Göttingen by bringing the young Ludwig Prandtl (1875–1953) and the more senior Carl Runge (1856–1927), both from the nearby Hanover. A picture of Prandtl at his water tunnel around 1935 is shown in Figure 2.1.

Prandtl had studied mechanical engineering at the Technische Hochschule (TH, Technical University) in Munich in the late 1890s. In his studies he was deeply influenced by August Föppl (1854–1924), whose textbooks on technical mechanics became legendary. After finishing his studies as mechanical engineer in 1898, Prandtl became Föppl's assistant and remained closely related to him throughout his life, intellectually by his devotion to technical mechanics and privately as Föppl's son-in-law (Vogel-Prandtl, 1993). Under Föppl's supervision Prandtl wrote his doctoral dissertation on a problem of

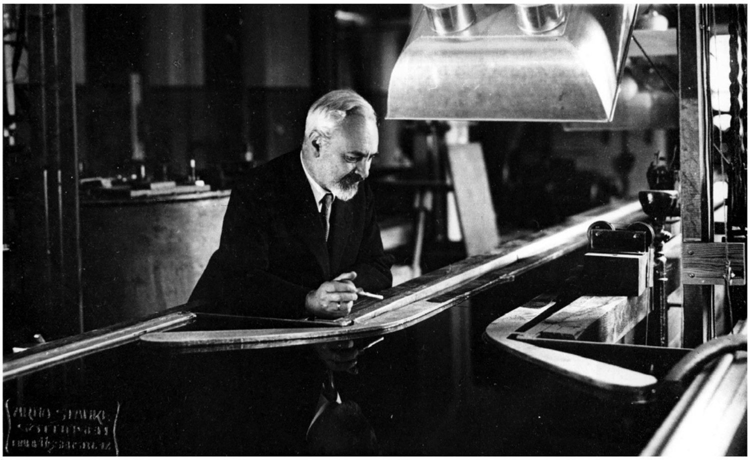


Figure 2.1 Ludwig Prandtl at his water tunnel in the mid to late 1930s. Reproduction from the original photograph DLR: FS-258.

technical mechanics (*Kipp-Erscheinungen, ein Fall von instabilem elastischem Gleichgewicht*: *On Tilting Phenomena, an Example of Unstable Elastic Equilibrium*). Technische Hochschulen were not then authorized to grant doctoral degrees, so that Prandtl had to perform the required academic rituals at the Philosophical Faculty of the neighboring Ludwig Maximilian University of Munich on 29 January 1900. At the time Technische Hochschulen were fighting a bitter struggle until they were granted equal rights with the Universities. The institutional schism affected in particular the academic disciplines at the interface of science and engineering, such as applied mathematics and technical mechanics (Oswatitsch and Wieghardt, 1987). On 1 January 1900, before receiving his doctorate, Prandtl had taken up an engineering position at the Maschinenbaugesellschaft in Nuremberg, which had just merged with Maschinenfabriken Augsburg to become MAN (Maschinenfabrik Augsburg Nürnberg: Machine Works of Augsburg and Nuremberg). At MAN he was first introduced to problems in fluid dynamics through designing a blower. Very shortly thereafter, he received an offer for the Chair of Mechanics at the Technische Hochschule Hanover. He left Nuremberg on 30 September 1901 to become at age 26 the youngest professor in Prussia (Vogel-Prandtl, 1993). In 1904 Felix Klein was able to convince Prandtl to take a non-full professor position at Göttingen University to become Head of the Department of Technical Physics at the Institute of Physics with the prospect of a co-directorship of a new Institute for Applied Mathematics and Mechanics. In the same vein, Klein had arranged Runge's call to Göttingen. In autumn 1905, Klein's institutional plans materialized. Göttingen University opened a new Institute for Applied Mathematics and Mechanics under the joint directorship of Runge and Prandtl. Klein also involved Prandtl as Director in the planning of an extramural Aerodynamic Research Institute, the Motorluftschiffmodell-Versuchsanstalt, which started its operation with the first Göttingen design windtunnnel in 1907 (Rotta, 1990; Oswatitsch and Wieghardt, 1987). Klein regarded Prandtl's "strong power of intuition and great originality of thought with the expertise of the engineer and the mastery of the mathematical apparatus" (Manegold, 1970, p. 232) ideal qualities for what he had planned to establish at Göttingen.

With these institutional measures, the stage was set for Prandtl's unique career between science and technology – and for the foundation of an academic school with a strong focus on basic fluid dynamics and their applications. Prandtl directed the Institut für Angewandte Mechanik of Göttingen University, the Aerodynamische Versuchsanstalt (AVA), as the rapidly expanding Motorluftschiffmodell-Versuchsanstalt (airship model test facility) was renamed after the First World War, and, after 1925, the associated Kaiser-Wilhelm-Institut (KWI) für Strömungsforschung. His ambitions and the history leading to the establishment of the KWI are well summarized in his opening speech at his institute, which has been translated into English (Prandtl, 1925E).

During the half century of Prandtl's Göttingen period, from 1904 until his death, his school extended Göttingen's fame from mathematics to applied mechanics, a specialty which acquired in this period the status of a self-contained discipline. Prandtl had more than eighty doctoral students, among them Heinrich Blasius, Theodore von Kármán, Max Munk, Johann Nikuradse, Walter Tollmien, Hermann Schlichting, Karl Wieghardt, and others who, like Prandtl, perceived fluid mechanics in general, and turbulence in particular, as a paramount challenge to bridge the gulf between theory and practice. Like Prandtl's institutional affiliations, his approach towards turbulence reflects a broad spectrum of 'pure' and 'applied' research (if such dichotomies make sense in turbulence research). We have to consider the circumstances and occasions in these settings in order to better characterize the approach of the Göttingen school on turbulence.

2.2 The boundary layer concept, 1904–1914

When Prandtl arrived in Göttingen in autumn 1904, he came with an asset: the boundary layer concept (Eckert, 2006, chapter 2; Meier, 2006). He was led to this concept during his short industrial occupation when he tried to account for the phenomenon of flow separation in diverging ducts. Prandtl presented the concept together with photographs of flow around obstacles in a water trough at the Third International Congress of Mathematicians in Heidelberg in August 1904 (Prandtl, 1905). In a summary, prepared at the request of the American Mathematical Society, he declared[1] that the "most important result" of this concept was that it offered an "explanation for the formation of discontinuity surfaces (vortex sheets) along continuously curved boundaries". In his Heidelberg presentation he expressed the same message in these words: "A fluid layer set in rotational motion by the friction at the wall moves into the free fluid and, exerting a complete change of motion, plays there a similar role as Helmholtz' discontinuity sheets" (Prandtl, 1905, p. 578). (For more on the emergence of Helmholtz's concept of discontinuity surfaces, see Darrigol, 2005, chapter 4.3).

According to the recollection of one participant at the Heidelberg congress, Klein recognized the momentousness of Prandtl's method immediately (Sommerfeld, 1935). However, if this recollection from many years later may be trusted, Klein's reaction was exceptional. The boundary layer concept required elaboration before its potential was more widely recognized (Dryden, 1955; Goldstein, 1969; Tani, 1977; Grossmann et al., 2004). Its modern understanding in terms of singular perturbation theory (O'Malley Jr., 2010) emerged only decades later. The first tangible evidence that Prandtl's concept provided more than qualitative ideas was offered by Blasius, who derived in his doctoral dissertation the coefficient for (laminar) skin friction from the boundary layer equations for the flow along a flat plate ('Blasius flow': Blasius, 1908; Hager, 2003). However, this achievement added little understanding to what Prandtl had considered the most important result of his concept, namely how vortical motion, to say nothing of turbulence, is created at the boundary.

Even before Prandtl arrived in Göttingen, the riddle of turbulence was a recurrent theme in Klein's lectures and seminars. In a seminar on hydraulics in the winter semester 1903/04 Klein called it a "true need of our time to bridge the gap between separate developments". The notorious gulf between hydraulics and hydrodynamics served to illustrate this need with many examples. The seminar presentations were expected to focus on the comparison between theory and experiment in a number of specific problems with the flow of water, such as the outflow through an orifice, the flow over a weir, pipe

[1] Undated draft in response to a request from 13 August 1904, Blatt 43, Cod. Ms. L. Prandtl 14, Acc. Mss. 1999.2, SUB.

flow, waves, the water jump ('hydraulic jump'), or the natural water flow in rivers.[2]

In the winter semester 1907/08 Klein dedicated another seminar to fluid mechanics, this time with the focus on 'Hydrodynamics, with particular emphasis of the hydrodynamics of ships'. With Prandtl and Runge as co-organizers, the seminar again involved a broad spectrum of problems from fluid mechanics that Klein and his colleagues regarded as suitable for mathematical approaches.[3] Theodore von Kármán, who made then his first steps towards an outstanding career in Prandtl's institute, presented a talk on unsteady potential motion. Blasius, who was finishing his dissertation on the laminar boundary layer in 1907, reviewed in two sessions contemporary research on turbulent flows. Other students and collaborators of Prandtl dealt with vortical motion (Karl Hiemenz) and boundary layers and the detachment of vortices (Georg Fuhrmann). Although little was published about these themes at the time, Klein's seminar served as a testing ground for debates on the notorious problems of fluid mechanics like the creation of vorticity in ideal fluids ('Klein's Kaffeelöffelexperiment': see Klein, 1910; Saffman, 1992, chapter 6).

With regard to turbulence, the records of Blasius' presentation from this seminar illustrate what Prandtl and his collaborators must have regarded as the main problems at that time. After reviewing the empirical laws, such as Chezy's law for channel flow and Reynolds' findings about the transition to turbulence in pipe flow (for these and other pioneering 19th-century efforts, see Darrigol, 2005, chapter 6), Blasius concluded that the problems "addressed to hydrodynamics" should be sorted into two categories: I. Explanation of instability; and II. Description of turbulent motion. These had to address the dichotomy of hydraulic description versus rational hydrodynamic explanations. Concerning the first category, the onset of turbulence, Blasius reviewed Hendrik Antoon Lorentz's recent approach where a criterion for the instability of laminar flow was derived from a consideration of the energy added to the flow by a superposed fluctuation (Lorentz, 1897, 1907). With regard to the second category, fully developed turbulence, Blasius referred mainly to Boussinesq's pioneering work (Boussinesq, 1897) where the effect of turbulence was described as an additional viscous term in the Navier–Stokes equation. In contrast to the normal viscosity, this additional 'turbulent' viscosity term was due to the exchange of momentum by the eddying motion in turbulent flow.

[2] Klein, handwritten notes. SUB Cod. Ms. Klein 19 E (Hydraulik, 1903/04), and the seminar protocol book, no. 20. Göttingen, Lesezimmer des Mathematischen Instituts. Available online at librarieswithoutwalls.org/klein.html.

[3] Klein's seminar protocol book, no. 27. Göttingen, Lesezimmer des Mathematischen Instituts. Available online at librarieswithoutwalls.org/klein.html.

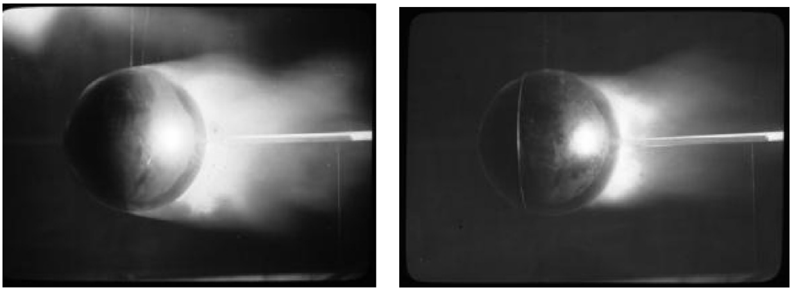
Boussinesq's concept had already been the subject of the preceding seminar in 1903/04, where the astronomer Karl Schwarzschild and the mathematicians Hans Hahn and Gustav Herglotz reviewed the state of turbulence (Hahn et al., 1904). However, the efforts in the seminar to determine the (unknown) eddy viscosity of Boussinesq's approach proved futile. "Agreement between this theory and empirical observations is not achieved," Blasius concluded in his presentation.[4]

In spite of the emphasis on the riddles of turbulence in these seminars, it is commonly reported that Prandtl ignored turbulence as a research theme until many years later. For example, the editors of his *Collected Papers* dated his first publication in the category *Turbulence and Vortex Formation* to the year 1921 (see below). The preserved archival sources, however, belie this impression. Prandtl started to articulate his ideas on turbulence much earlier. "Turbulence I: Vortices within laminar motion", he wrote on an envelope with dozens of loose manuscript pages. The first of these pages is dated by himself as 3 October 1910, with the heading *Origin of turbulence*. Prandtl considered there "a vortex line in the boundary layer close to a wall" and argued that such a vortical motion "fetches (by frictional action) something out of the boundary layer which, because of the initial rotation, becomes rolled up to another vortex which enhances the initial vortex." Thus he imagined how flows become vortical due to processes that originate in the initially laminar boundary layer.[5] In the same year he published a paper on *A relation between heat exchange and flow resistance in fluids* (Prandtl, 1910) which extended the boundary layer concept to heat conduction. Although it did not explicitly address turbulence – the article is more renowned because Prandtl introduced here what was later called the 'Prandtl number' – it reveals Prandtl's awareness for the differences of laminar and turbulent flow with regard to heat exchange and illustrates from a different perspective how turbulence entered Prandtl's research agenda (Rotta, 2000).

Another opportunity to think about turbulence from the perspective of the boundary layer concept came in 1912 when wind tunnel measurements about the drag of spheres displayed discrepant results. When Otto Föppl (1885–1963), Prandtl's brother-in-law and collaborator at the airship model test facility, compared the data from his own measurements in the Göttingen wind tunnel with those from the laboratory of Gustave Eiffel (1832–1921) in Paris, he found a blatant discrepancy and supposed that Eiffel or his collaborator had omitted a factor of 2 in the final evaluation of their data (Föppl, 1912). Provoked by this claim, Eiffel performed a new test series and found that

[4] Klein's seminar protocol book, no. 27, p. 80. Göttingen, Lesezimmer des Mathematischen Instituts. Available online at http://www.librarieswithoutwalls.org/felixKlein.html.

[5] Cod. Ms. L. Prandtl 18 (*Turbulenz I: Wirbel in Laminarbewegung*), Acc. Mss. 1999.2, SUB.



(a) Without a trip wire. (b) With a trip wire.

Figure 2.2 Turbulence behind a sphere made visible with smoke. Reproduction from the original 1914 photograph. GOAR: GK-0116 and GK-0118.

the discrepancy was not the result of an erroneous data evaluation but a new phenomenon which could be observed only at higher air speeds than those attained in the Göttingen wind tunnel (Eiffel, 1912). After inserting a nozzle into their wind tunnel, Prandtl and his collaborators were able to reproduce Eiffel's discovery: at a critical air speed the drag coefficient suddenly dropped to a much lower value. Prandtl also offered an explanation of the new phenomenon. He assumed that the initially laminar boundary layer around the sphere becomes turbulent beyond a critical air speed. On the assumption that the transition from laminar to turbulent flow in the boundary layer is analogous to Reynolds' case of pipe flow, Prandtl displayed the sphere drag coefficient as a function of the Reynolds number, UD/ν (flow velocity U, sphere diameter D, kinematic viscosity ν), rather than, as did Eiffel, of the velocity; thus he demonstrated that the effect occurred at roughly the same Reynolds number even if the individual quantities differed widely (the diameters of the spheres ranged from 7 to 28 cm; the speed in the wind tunnel was varied between 5 and 23 m/s). Prandtl further argued that the turbulent boundary layer flow entrains fluid from the wake so that the boundary layer stays attached to the surface of the sphere longer than in the laminar case. In other words, the onset of turbulence in the boundary layer reduces the wake behind the sphere and thus also its drag. But the argument that turbulence decreases the drag seemed so paradoxical that Prandtl conceived an experimental test: when the transition to turbulence in the boundary layer was induced otherwise, e.g. with a thin 'trip wire' around the sphere or a rough surface, the same phenomenon occured. When smoke was added to the air stream, the reduction of drag became visible by the reduced extension of the wake behind the sphere (Wieselsberger, 1914; Prandtl, 1914) (see Figure 2.2).

2.3 A working program for a theory of turbulence

During the First World War turbulence became pertinent in many guises. Arnold Sommerfeld (1868–1951), theoretical physicist at the University of Munich, once forwarded to Prandtl a request “concerning the fall of bombs in water and air”. Sommerfeld was involved at that time in other war-related research (about wireless telegraphy) and had heard about this problem from a Major whom he had met during a visit in Berlin. “It deals with the drag of a sphere (radius a) moving uniformly through water (density ρ) at a velocity V. At Re (Reynolds number) > 1000 the drag is $W = \psi\rho a^2 V^2$.” By similarity, “ψ should be universal and also independent of the fluid”, Sommerfeld alluded to Prandtl’s recent study about the drag of spheres in air; but according to older measurements of the friction coefficient ψ for water this was not the case.[6] Prandtl suspected an error with the assessment of the experimental measurements. Furthermore, the impact of a falling bomb on the water surface involved additional effects so that a comparison was difficult.[7] A few months later, the aerodynamics of bomb shapes became officially part of Prandtl’s war work.[8]

The turbulence effect as observed with the drag of spheres became also pertinent for the design of airplanes. The struts and wires which connected the wings of bi- and triplanes were subject to the same sudden changes of drag. For this reason, Prandtl’s institute was charged with a systematic wind tunnel investigation of struts and wires. The goal was to find out how the sudden change of drag could be avoided by choosing appropriate strut and wire shapes. “The critical range [of Reynolds numbers] is considered as the interval within which there are two fundamentally different modes of flow”, Prandtl’s collaborator, Max Munk, explained in a technical war report on measurements of the drag of struts. The report also mentioned how this phenomenon occurred in practice. In particular, a reduction of the speed, for example, when the plane changes from horizontal flight into a climb, results in a sudden increase of the drag coefficient, and often of a considerable increase of the drag itself. It was therefore not sufficient to minimize the drag by streamlining the profile of a strut, but also to give it a shape that did not experience the sudden change of drag when the airplane passed through the critical speed range (Munk, 1917).

[6] Sommerfeld to Prandtl, 9 May 1915. GOAR 2666.
[7] Prandtl to Sommerfeld, 14 May 1915. GOAR 2666.
[8] He received, for example, contracts from the Bombenabteilung der Prüfanstalt u. Werft der Fliegertruppen, dated 23 December 1915, concerning ‘Fliegerbombe, M 237’, and on ‘Carbonit-Bomben, Kugelform’, dated 1 September 1916. GOAR 2704B.

In view of such practical relevance, Prandtl sketched[9] in March 1916 a *Working program about the theory of turbulence*. Like Blasius in his presentation in Klein's seminar, Prandtl discriminated between the onset of turbulence, i.e. the transition from laminar to turbulent flow, and what he called accomplished turbulence, i.e. fully developed turbulence, as the two pillars of this research program. The onset of turbulence was generally perceived as the consequence of a hydrodynamic instability, a problem with a long history of frustrated efforts (Darrigol, 2005); although it had been revived during the preceding decade by William McFadden Orr, Sommerfeld, Ludwig Hopf, Fritz Noether and others, a solution seemed out of sight (Eckert, 2010). Prandtl sketched plane flows with different piece-wise linear velocity profiles. The stability of such flow configurations had been extensively studied since the 1880s by Lord Rayleigh for the inviscid case. Profiles with an inflection were unstable according to Rayleigh's (1887) analysis. Prandtl's strategy seemed clear: he approached the stability analysis from the limiting case of infinite Reynolds numbers, i.e. the inviscid case treated by Rayleigh, in order to derive from this limit approximations for flows of low viscosity. Like his boundary layer concept, this approach would be restricted to high Reynolds numbers (unlike the Orr–Sommerfeld approach which applied to the full range of Reynolds numbers). According to his sketches and somewhat cryptic descriptions, Prandtl attempted to study the behavior of "a sinusoidal discontinuity" in a "stripe flow". Prandtl's 'stripes,' i.e. piece-wise linear flow profiles, indicate that he aimed at a theory for the onset of turbulence in the plane flow bounded by two fixed walls and a flow bounded by a single wall. The latter configuration obviously was perceived as an approximation of the 'Blasius flow', i.e. the velocity profile of the laminar boundary layer flow along a flat plate. According to Rayleigh's inflection theorem, both flows were stable in the inviscid case because the curvature of the velocity profile did not change direction. The focus was on the boundary layer motion with Rayleigh oscillation, as Prandtl concluded this part of his turbulence program.

With regard to fully developed turbulence, the other part of his working program, Prandtl apparently had no particular study in mind as a starting point. "Statistical equilibrium of a set of vortices in an ideal fluid in the vicinity of a wall", he noted as one topic for future research. For the goal of a "complete approach for very small friction" he started from the assumption that vortices from the wall ("in the boundary layer") are swept into the fluid by "disordered motion". He envisioned a balance between the vortex creation at the wall and the vortices destroyed in the fluid as a result of friction. For a closer analysis

[9] Page 15 (dated 6 March 1916) in Cod. Ms. L. Prandtl, 18, Acc. Mss. 1999.2, SUB.

of the involved vortex interaction he introduced what he called the rough assumption that the vortex remains unchanged for a certain time $\sim r^2/\nu$ and then suddenly disappears, whereby it communicates its angular momentum to the mean flow.[10]

During the war Prandtl had more urgent items on his agenda (Rotta, 1990, pp. 115–193). But the riddle of turbulence as a paramount challenge did not disappear from his mind. Nor from that of his former student, the prodigy Theodore von Kármán, who returned after the War to the Technische Hochschule Aachen as Director of the then fledgling Aerodynamic Institute. Both the Aachen and the Göttingen fluid dynamicists pursued the quest for a theory of turbulence in a fierce rivalry. "The competition was gentlemanly, of course. But it was first-class rivalry nonetheless," Kármán later recalled, "a kind of Olympic Games, between Prandtl and me, and beyond that between Göttingen and Aachen" (von Kármán, 1967, p. 135). Since they had nothing published on turbulence, both Prandtl and Kármán pondered how to ascertain their priority in this quest. In summer 1920, Prandtl supposed that von Kármán used a forthcoming science meeting in Bad Nauheim to present a paper on turbulence at this occasion. "I do not yet know whether I can come", he wrote[11] to his rival, "but I wish to be oriented about your plans. As the case may be I will announce something on turbulence (experimental) as well. I have now visualized turbulence with lycopodium in a 6 cm wide channel." The Aachen–Göttingen rivalry had not yet surfaced publicly at this occasion. By correspondence, however, it was further developing. The range of topics encompassed Prandtl's entire working program. Early in 1921 Prandtl learned that von Kármán was busy elaborating a theory of fully developed turbulence in the boundary layer along a flat wall – with "fabulous agreement with observations". Ludwig Hopf and another collaborator of the Aachen group had by this time started with hot-wire experiments. Hopf revealed[12] that in Aachen they planned to measure in a water channel the mean square fluctuation and the spectral distribution of the fluctuations.

Little seems to have resulted from these experiments, neither in Aachen by means of the hot-wire technique nor in Prandtl's laboratory by visualizing turbulence with lycopodium. Von Kármán's theoretical effort, however, appeared promising. "Dear Master, colleague, and former boss", Kármán addressed Prandtl in a five-page letter with ideas for a turbulent boundary layer

[10] Page 16 in Cod. Ms. L. Prandtl, 18, Acc. Mss. 1999.2, SUB. Apparently r and ν are the radius of the vortex and the kinematic viscosity of the fluid, respectively. Prandtl did not define the quantities involved here. His remarks are rather sketchy and do not lend themselves for a precise determination of the beginnings of his future mixing length approach.

[11] Prandtl to Kármán, 11 August 1920. GOAR 1364.

[12] Hopf to Prandtl, 3 February 1921. MPGA, Abt. III, Rep. 61, Nr. 704.

theory (see below) and about the onset of turbulence.[13] The latter was regarded as *the* turbulence problem. The difficulty in explaining the transition from laminar to turbulent flow had been rated as a paramount riddle since the late 19th century. In his dissertation performed under Sommerfeld in 1909, Ludwig Hopf had titled the introductory section *The turbulence problem*, because neither the energy considerations of Reynolds and Lorentz nor the stability approaches of Lord Kelvin and Lord Rayleigh were successful. Hopf was confronted with the problem in the wake of Sommerfeld's own stability approach to viscous flows, but "the consequent analysis of the problem according to the method of small oscillations by Sommerfeld is not yet accomplished" (Hopf, 1910, pp. 6–7). In the decade that followed the problem was vigorously attacked by this technique (later labeled as the Orr–Sommerfeld method) – with the discrepant result that plane Couette flow seemed stable for all Reynolds numbers (Eckert, 2010).

In comparison with these efforts, Prandtl's approach as sketched in his working program appeared like a return to the futile attempts of the 19th century: "At large Reynolds number the difference between viscous and inviscid fluids is certainly imperceptible," Hopf commented[14] on Prandtl's idea to start from the inviscid limit, but at the same time he regarded it "questionable whether one is able to arrive at a useful approximation that leads down to the critical number from this end". In response to such doubts Prandtl began to execute his working program about the onset of turbulence in plane flows with piecewise linear flow profiles. "Calculation according to Rayleigh's papers III, p. 17ff," he noted on a piece of paper in January 1921, followed by several pages of mathematical calculations.[15] Despite their initial reservations, the Aachen rivals were excited about Prandtl's approach. Von Kármán immediately rushed his collaborators to undertake a stability analysis for certain piecewise linear flow profiles, Hopf confided to Prandtl.[16] Prandtl had by this time already asked a doctoral student to perform a similar study. "Because it deals with a doctoral work, I would be sorry if the Aachener would publish away part of his dissertation," he asked Hopf, so as not to interfere in this effort. Von Kármán responded that the Aachen stability study was aiming at quite different goals, namely the formation of vortices in the wake of an obstacle (labeled later as the 'Kármán vortex street' after von Kármán's earlier theory about this phenomenon; Eckert, 2006, chapter 2). The new study was motivated by "the hope to determine perhaps the constants that have been left indetermined in my old theory," wrote Kármán, attempting to calm Prandtl's worry. Why not arrange

[13] Kármán to Prandtl, 12 February 1921. GOAR 3684.
[14] Hopf to Prandtl, 27 October 1919. GOAR 3684.
[15] Pages 22–26 in Cod. Ms. L. Prandtl, 18, Acc. Mss. 1999.2, SUB.
[16] Hopf to Prandtl, 3 February 1921. MPGA, III, Rep. 61, Nr. 704.

a division of labor between Göttingen and Aachen, he further suggested[17], so that his group would deal with these wake phenomena and Prandtl's doctoral student with boundary layer instability. Prandtl agreed and suggested that von Kármán should not feel hemmed in by his plans. He explained[18] once more that the focus at Göttingen was to study the onset of turbulence in the boundary layer. "We have now a method for approximately taking into account friction."

A few months later Prandtl reported that the calculations of his doctoral student were terribly complicated and yielded "a peculiar and unpleasant result". If the corresponding flow is unstable according to Rayleigh's inviscid theory, the instability was not reduced by taking viscosity into account – as they had expected – but *increased*. The calculation was done in first-order approximation, but its extension to the second-order seemed almost hopeless, Prandtl wrote[19] in frustration, "and so, once more, we do not obtain a critical Reynolds number. There seems to be a very nasty devil in turbulence so that all mathematical efforts are doomed to failure."

At this stage Prandtl published his and his doctoral student's, Oskar Tietjens (1893–1971), effort. In addition to the profiles which were unstable in the inviscid case, the study was extended to those that were stable in the inviscid case (i.e. ones without an inflection) – with the surprising result that these profiles also became unstable if viscosity was included. Contrary to the stability deadlock of the earlier studies concerning plane Couette flow, Prandtl's approach left the theory in an instability deadlock. "We did not want to believe in this result and have performed the calculation three times independently in different ways. There was always the same sign which indicated instability" (Prandtl, 1921a, p. 434).

Prandtl's paper appeared in a new journal edited by the applied mathematician Richard von Mises, the *Zeitschrift für Angewandte Mathematik und Mechanik* (*ZAMM*) where the turbulence problem was presented as a major challenge. In his editorial von Mises described the the present state of the theory as completely open. He regarded it as undecided whether the viscous flow approach is able to explain turbulence at sufficient mathematical depth (von Mises, 1921, p. 12). Fritz Noether, like Hopf, a Sommerfeld disciple who had struggled with this matter for years, introduced the subject with a review article titled *The Turbulence Problem* (Noether, 1921). He summarized the series of futile attempts of the preceding decades and presented the problem in a generic manner. (Noether presented the 'stability equation' or 'perturbation differential equation' – to quote the contemporary designations – in the form in which it became familiar later as the Orr–Sommerfeld equation. His paper

17 Kármán to Prandtl, 12 February 1921. GOAR 3684.
18 Prandtl to Kármán, 16 February 1921. MPGA, III, Rep. 61, Nr. 792.
19 Prandtl to Kármán, 14 June 1921. MPGA, III, Rep. 61, Nr. 792.

became the door-opener for many subsequent studies of the Orr–Sommerfeld approach.) Noether was also well informed about the Göttingen effort, as is evident from his correspondence with Prandtl. In one of his letters[20] he expressed some doubts about Prandtl's approach, but he belittled his dissent and regarded it merely as a difference of mindset and expression. Another contributor to the turbulence problem in this first volume of *ZAMM* was Ludwig Schiller, a physicist working temporarily in Prandtl's laboratory; Schiller (1921) surveyed the experimental efforts at measuring the onset of turbulence.

The turbulence problem was also discussed extensively in September 1921 at a conference in Jena, where the Deutsche Physikalische Gesellschaft, the Deutsche Gesellschaft für Technische Physik and the Deutsche Mathematiker-Vereinigung convened their annual meetings of that year in a common event. At this occasion, Prandtl's *Remarks about the Onset of Turbulence* caused quite a stir. "With regard to the theoretical results which have always yielded stability it should be noted that these referred to the so-called Couette case," said Prandtl, explaining the difference between his result and the earlier studies. But Sommerfeld found it "very strange and at first glance unlikely" that all flows are unstable except Couette flow. "What causes the special position of Couette flow?" asked Sommerfeld. Kármán pointed to kinks at arbitrary positions of the piecewise linear profiles as a source of arbitrariness. Hopf criticized Prandtl's approximation $Re \to \infty$ (Prandtl, 1922, pp. 22–24).

The Jena conference and the articles on the turbulence problem in *ZAMM* from the year 1921 marked the beginning of a new period of research on the onset of turbulence. From then on Prandtl did not participate with his own contributions to this research. But he continued to supervise doctoral dissertations about this part of his working program. Tietjens (1925) paved the way along which Walter Tollmien (1900–1968), in another Göttingen doctoral dissertation, achieved the first complete solution of the Orr–Sommerfeld equation for a special flow (Tollmien, 1929). A few years later, another disciple of Prandtl, Hermann Schlichting (1907–1982), further extended this theory (Schlichting, 1933), so that the process of instability could be analysed in more detail. But the 'Tollmien–Schlichting' approach remained disputed until it was experimentally corroborated in World War II (Eckert, 2008).

2.4 Skin friction and turbulence I: the 1/7th law

Originally, Prandtl's boundary layer concept had focused on laminar flow. Ten years later, with the interpretation of Eiffel's drag phenomenon as a turbulence

[20] Noether to Prandtl, 29 June 1921. GOAR 3684.

effect, boundary layer flow could also be imagined as fully turbulent. From a practical perspective, the latter appeared much more important than the former. Data on fluid resistance in pipes, as measured for decades in hydraulic laboratories, offered plenty of problems for testing theories about turbulent friction. Blasius, who had moved in 1911 to the Berlin Testing Establishment for Hydraulics and Ship Building (Versuchsanstalt für Wasserbau und Schiffbau), published in 1913 a survey of pipe flow data: when displayed as a function of the Reynolds number *Re*, the coefficient for 'hydraulic' (i.e. turbulent) friction varied in proportion to $Re^{-1/4}$ (in contrast to laminar friction at low Reynolds numbers, where it is proportional to Re^{-1}) (Blasius, 1913).

No theory could explain this empirical 'Blasius law' for turbulent pipe flow. But it could be used to derive other semi-empirical laws, such as the distribution of velocity in the turbulent boundary layer along a plane smooth wall. When Kármán challenged Prandtl in 1921 with the outline of such a theory, he recalled that Prandtl had told him earlier how one could extrapolate from pipe flow to the flow along a plate, and that Prandtl already knew that the velocity distribution was proportional to $y^{1/7}$, where y was the distance from the wall. Prandtl responded that he had known this "already for a pretty long time, say since 1913". He claimed that he had already in earlier times attempted to calculate boundary layers in which he had assumed a viscosity enhanced by turbulence, which he chose for simplicity as proportional to the distance from the wall and proportional to the velocity in the free flow. But he admitted that Kármán had advanced further with regard to a full-fledged turbulent boundary layer theory. "I have planned something like this only for the future and have not yet begun with the elaboration." Because he was busy with other work he suggested[21] that Kármán should proceed with the publication of this theory: "I will see afterwards how I can gain recognition with my different derivation, and I can get over it if the priority of publishing has gone over to friendly territory."

Kármán published his derivation without further delay in the first volume of *ZAMM* (von Kármán, 1921) with the acknowledgement that it resulted from a suggestion "by Mr. Prandtl in an oral communication in Autumn 1920". Prandtl's derivation appeared in print only in 1927 – with the remark that "the preceding treatment dates back to Autumn 1920" (Prandtl, 1927a). Johann Nikuradse (1894–1979), whom Prandtl assigned by that time an experimental study about the velocity distribution in turbulent flows as subject of a doctoral work, dated Prandtl's derivation more precisely to a discourse in Göttingen on 5 November, during the winter semester of 1920 (Nikuradse,

[21] Prandtl to Kármán, 16 February 1921. MPGA, Abt. III, Rep. 61, Nr. 792.

1926, p. 15). Indeed, Prandtl outlined this derivation in notices dated (by himself) to 28 November 1920.[22] Further evidence is contained in the first volume of the *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, accomplished at "Christmas 1920" (according to the preface), where Prandtl offered a formula for the friction coefficient proportional to $Re^{-1/5}$, with the Reynolds number Re related to the length of the plate (Prandtl, 1921b, p. 136). Although Prandtl did not present the derivation, he could not have arrived at this friction coefficient without the 1/7th law for the velocity distribution. (The derivation was based on the assumption that the shear stress at the wall inside the tube only depends on the flow in the immediate vicinity of the wall; hence it should not depend on the radius of the tube. Under the additional assumption that the velocity grows according to a power law with increasing distance from the wall, the derivation was straightforward.)

Kármán presented his theory on turbulent skin friction again in 1922 at a conference in Innsbruck (von Kármán, 1924). He perceived it only as a first step on the way towards a more fundamental understanding of turbulent friction. The solution, he speculated at the end of his Innsbruck talk, would probably come from a statistical consideration. But in order to pursue such an investigation "a fortunate idea" was necessary, "which so far has not yet been found" (von Kármán, 1924, p. 167). (For more on the quest for a statistical theory in the 1920s, see Battimelli, 1984.) Prandtl, too, raised little hope for a more fundamental theory of turbulence from which empirical laws, such as that of Blasius, could be derived from first principles: "You ask for the theoretical derivation of Blasius' law for pipe friction," Prandtl responded[23] to the question of a colleague in 1923. "The one who will find it will thereby become a famous man!"

2.5 The mixing length approach

Prandtl's ideas concerning fully developed turbulence remained the subject of informal conversations and private correspondence for several more years after 1921. "I myself have brought nothing to paper concerning the 1/7-law," Prandtl wrote[24] to Kármán in continuation of their correspondence about the turbulent boundary layer theory in summer 1921. A few years later, the velocity distribution in the turbulent boundary layer of a smooth plate in a wind tunnel was measured directly in Johannes M. Burgers' (1895–1981) laboratory in Delft

[22] Prandtl, notices, MPGA, Abt. III, Rep. 61, Nr. 2296, page 65.
[23] Prandtl to Birnbaum, 7 June 1923. MPGA, Abt. III, Rep. 61, Nr. 137.
[24] Prandtl to Kármán, 14 June 1921. MPGA, Abt. III, Rep. 61, Nr. 792.

by the new method of hot wire anemometry (Burgers, 1925). Prandtl had hesitated in 1921 to publish his derivation of the 1/7th law because, as he revealed in another letter[25] to his rival at Aachen, he aimed at a theory in which the experimental evidence would play a crucial role. Four years later, with the data from Burgers' laboratory, from the dissertation of Nikuradse (1926), and from other investigations about the resistance of water flow in smooth pipes (Jakob and Erk, 1924), this evidence was available. The experiments confirmed the 1/7th law within the range of Reynolds numbers for which the Blasius 1/4th law was valid. But they raised doubts whether it was valid for higher Reynolds numbers. Prandtl, therefore, attempted to generalize his theoretical approach so that he could derive from any empirical resistance law a formula for the velocity distribution. He wrote[26] to von Kármán in October 1924 thus:

> I myself have occupied myself recently with the task to set up a differential equation for the mean motion in turbulent flow, which is derived from rather simple assumptions and seems appropriate for very different cases. . . . The empirical is condensed in a length which is entirely adjusted to the boundary conditions and which plays the role of a free path length.

Thus he alluded to the 'mixing length' approach, as it would be labeled later. He published this approach together with the derivation of the 1/7th law. Prandtl's basic idea was to replace the unknown eddy viscosity ϵ in Boussinesq's formula for the turbulent shear stress, $\tau = \rho\epsilon\frac{dU}{dy}$, by an expression which could be tested by experiments. The dimension of ϵ is $\frac{\mathrm{m}^2}{\mathrm{s}}$, i.e. the product of a length and a velocity. Prandtl made the Ansatz

$$\epsilon = l \cdot l \left|\frac{dU}{dy}\right|,$$

with $l\left|\frac{dU}{dy}\right|$ as the mean fluctuating velocity with which a 'Flüssigkeitsballen' ('fluid bale' or 'fluid eddy') caused a lateral exchange of momentum. Formally, the approach was analogous to the kinetic theory of gases, where a particle could travel a mean free path length before it exchanged momentum with other particles. In turbulent flow, however, the exchange process was less obvious. Prandtl visualized l first as a braking distance (Prandtl, 1925, p. 716, or, in English, Prandtl, 1949E) then as a mixing length (Prandtl, 1926a, p. 726). He made this approach also the subject of his presentation at the Second International Congress of Applied Mechanics, held in Zürich during 12–17 September 1926 (Prandtl, 1927b).

[25] Prandtl to Kármán, 16 February 1921. MPGA, Abt. III, Rep. 61, Nr. 792.
[26] Prandtl to Kármán, 10 October 1924. MPGA, Abt. III, Rep. 61, Nr. 792.

The historic papers on turbulent stress and eddy viscosity by Reynolds and Boussinesq were of course familiar to Prandtl since Klein's seminars in 1904 and 1907, but without further assumptions these approaches could not be turned into practical theories. At a first look, Prandtl had just exchanged one unknown quantity (ϵ) with another (l). However, in contrast to the eddy viscosity ϵ, the mixing length l was a quantity which, as Prandtl had written[27] to Kármán, "is entirely adjusted to the boundary conditions" of the problem under consideration. The problem of turbulent wall friction, however, required rather sophisticated assumptions about the mixing length. Without wall interactions, the mixing length could be adjusted less arbitrarily. Prandtl resorted to other phenomena for illustrating the mixing length approach, such as the mixing of a turbulent jet ejected from a nozzle into an ambient fluid at rest. In this case the assumption that the mixing length is proportional to the width of the jet in each cross-section gave rise to a differential equation from which the broadening of the jet behind the nozzle could be calculated. The theoretical distribution of mean flow velocities obtained by this approach was in excellent agreement with experimental measurements (Prandtl, 1927b; Tollmien, 1926).

For the turbulent shear flow along a wall, however, the assumption of proportionality between the mixing length l and the distance y from the wall did not yield the 1/7th law as Prandtl had hoped. Instead, when he attempted to derive the distribution of velocity for plane channel flow, he arrived at a logarithmic law – which he dismissed because of unpleasant behavior at the centerline of the channel (see Figure 2.3).[28] From his notes in summer 1924 it is obvious that he struggled hard to derive an appropriate distribution of velocity from one or another plausible assumption for the mixing length – and appropriate meant to him that the mean flow $U(y) \propto y^{1/7}$, not some logarithmic law.

Three years later (Prandtl, 1930, p. 794) in a lecture in Tokyo in 1929, he dismissed the logarithmic velocity distribution again. He argued that "l proportional y does not lead to the desired result because it leads to U prop. $\log y$, which would yield $U = -\infty$ for $y = 0$."

This provided an opportunity for Kármán to win the next round in their 'gentlemanly' competition.

2.6 Skin friction and turbulence II: the logarithmic law and beyond

In June 1928, Walter Fritsch, a student of Kármán, published the results of an experimental study of turbulent channel flow with different wall surfaces

27 Prandtl to Kármán, 10 October 1924. MPGA, Abt. III, Rep. 61, Nr. 792.
28 Prandtl, notices, MPGA, Abt. III, Rep. 61, Nr. 2276, page 12.

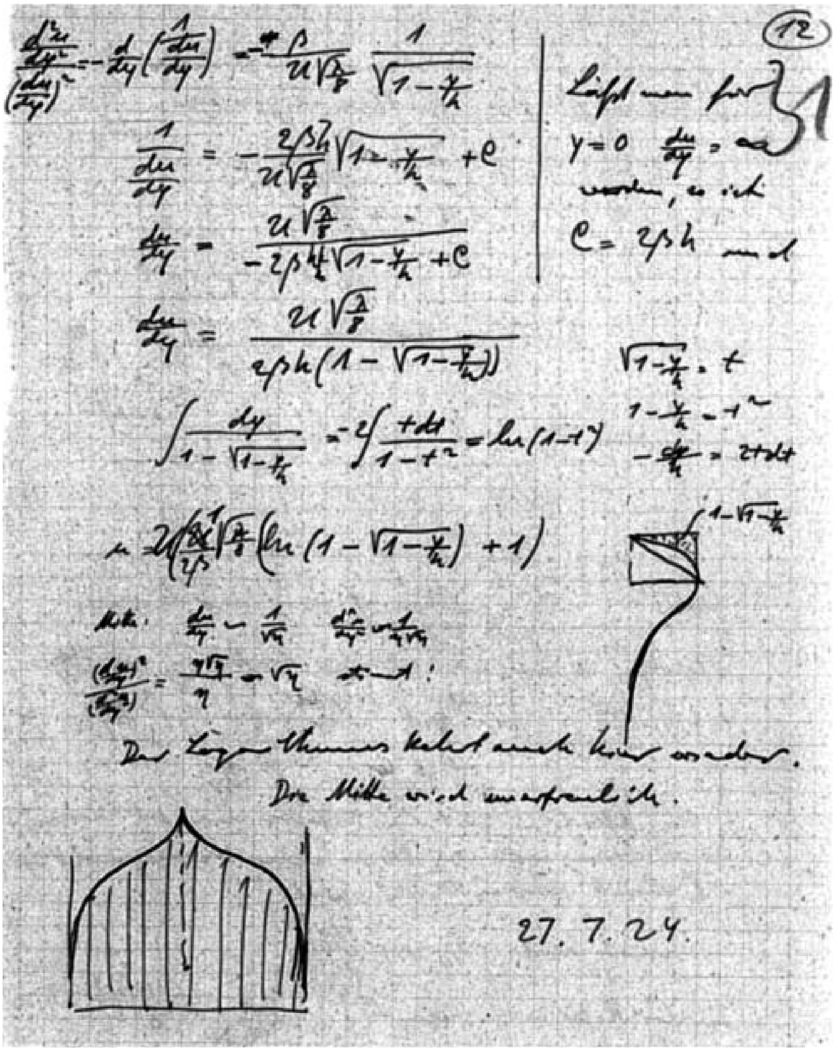


Figure 2.3 Excerpt of Prandtl's 'back of the envelope' calculations from 1924.

(Fritsch, 1928). He found that the velocity profiles line up with each other in the middle parts if they are shifted parallel. This suggested that the velocity distribution in the fluid depends only on the shear stress transferred to the wall and not on the particular wall surface structure. Kármán derived from this empirical observation a similarity approach. In a letter to Burgers he praised this approach for its simplicity: "The only important constant thereby is the proportionality factor in the vicinity of the wall." As a result, he was led to logarithmic laws both for the velocity distribution in the turbulent boundary layer and for the turbulent skin friction coefficient. "The resistance law fits very well with measurements in all known regions," he concluded, with a hint to recent measurements.[29]

The recent measurements to which Kármán alluded where those of Fritsch in Aachen and Nikuradse in Göttingen. The latter, in particular, showed a marked deviation from Blasius' law, and hence from the 1/7th law for the distribution of velocity, at higher Reynolds numbers. Nikuradse had presented some of his results in June 1929 at a conference in Aachen (Nikuradse, 1930); the comprehensive study appeared only in 1932 (Nikuradse, 1932). By introducing a dimensionless wall distance $\eta = v_* y/\nu$ and velocity $\varphi = u/v_*$, where $v_* = \sqrt{\tau_0/\rho}$ is the friction velocity, τ_0 the shear stress at the wall and ρ the density, Nikuradse's data suggested a logarithmic velocity distribution of the form $\varphi = a + b \log \eta$.

Backed by these results from Prandtl's laboratory, Kármán submitted a paper entitled *Mechanical Similarity and Turbulence* to the Göttingen Academy of Science. Unlike Prandtl, he introduced the mixing length as a characteristic scale of the fluctuating velocities determined by $l = kU'/U''$, where k is a dimensionless constant (later called the 'Kármán constant') and U', U'' are the first and second derivatives of the mean velocity of a plane parallel flow in the x-direction with respect to the perpendicular coordinate y. He derived this formula from the hypothesis that the velocity fluctuations are similar anywhere and anytime in fully developed turbulent flow at some distance from a wall. He had plane channel flow in mind, because he chose his coordinate system so that the x-axis coincided with the centerline between the walls at $y = \pm h$. The approach would fail both at the center line and at the walls, but was supposed to yield reasonable results in between. (For more detail on Kármán's approach, see Chapter 3 by Leonard and Peters.) Whereas Prandtl's approach required a further assumption about the mixing length, Kármán's l was an explicit function of y at any point in the cross-section of the flow. Kármán obtained a logarithmic velocity distribution and a logarithmic formula for the turbulent friction coefficient (von Kármán, 1930a).

A few months later, Kármán presented his theory at the Third International Congress of Applied Mechanics, held in Stockholm during 24–29 August 1930. For this occasion he also derived the resistance formula for the turbulent skin friction of a smooth plate. "The resistance law is no power law," hinting at the earlier efforts of Prandtl and himself. "I am convinced that the form of the resistance law as derived here is irrevocable." He presented a diagram about

[29] Kármán to Burgers, 12 December 1929. TKC 4.22.

the plate skin friction where he compared the 'Prandtl v. Kármán 1921' theory with the 'new' one, and with recent measurements from the Hamburgische Schiffbau–Versuchsanstalt. "It appears to me that for smooth plates the last mismatch between theory and experiment has disappeared," he concluded his Stockholm presentation (von Kármán, 1930b).

Prandtl was by this time preparing a new edition of the *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen* and eager to include the most recent results.[30] The practical relevance of Kármán's theory was obvious. In May 1932, the Hamburgische Schiffbau–Versuchsanstalt convened a conference where the recent theories and experiments about turbulent friction were reviewed. Kármán was invited for a talk on the theory of the fluid resistance, but he could not attend so that he contributed only in the form of a paper which was read by another attendee (von Kármán, 1932). Franz Eisner, a scientist from the Preussische Versuchsanstalt für Wasserbau und Schiffbau in Berlin, addressed the same theme from a broader perspective, and Günther Kempf from the Hamburg Schiffbau–Versuchsanstalt presented recent results about friction on smooth and rough plates (Eisner, 1932b; Kempf, 1932). Prandtl and others were invited to present commentaries and additions (Prandtl et al., 1932). By and large, this conference served to acquaint practitioners, particularly engineers in shipbuilding, with the recent advances achieved in the research laboratories in Göttingen, Aachen and elsewhere.

Two months after this conference, the Schiffbautechnische Gesellschaft published short versions of these presentations in its journal *Werft, Reederei, Hafen*. From Eisner's presentation a diagram about plate resistance was shown which characterized the logarithmic law "after Prandtl (Ergebnisse AVA Göttingen, IV. Lieferung 1932" as the best fit of the experimental values. According to this presentation, the "interregnum of power laws" had lasted until 1931, when Prandtl formulated the correct logarithmic law (Eisner, 1932a). When Kármán saw this article he was upset. He felt that his breakthrough for the correct plate formula in 1930 as he had presented it in Stockholm was ignored. He complained in a letter to Prandtl[31] that from the article about the Hamburg conference "it looks as if I have given up working on this problem after 1921, and that everything has been done in 1931/32 in Göttingen". He asked Prandtl to correct this erroneous view in the Göttingen *Ergebnisse*, which he regarded as the standard reference work for all future reviews. "I write so frankly how I think in this matter because I know you as the role model of a just man," appealing to Prandtl's fairness. But he had little sympathy for "your lieutenants who

[30] Prandtl to Kármán's colleagues at Aachen, 30 October 1930. TKC 23.43; Prandtl to Kármán, 29 November 1930; Kármán to Prandtl, 16 December 1930. MPGA, Abt. III, Rep. 61, Nr. 792.

[31] Kármán to Prandtl, 26 September 1932. MPGA, Abt. III, Rep. 61, Nr. 793.

understandably do not know other gods beside you. They wish to claim everything for Göttingen." He was so worried that he also sent Prandtl a telegram[32] with the essence of his complaint.

Prandtl responded immediately. He claimed[33] that he had no influence on the publications in *Werft, Reederei, Hafen*. With regard to the *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen* he calmed Kármán's worries saying that in the publication they would of course refer to the latter's papers. As in the preceeding volumes of the *Ergebnisse*, the emphasis was on experimental results. The news about the logarithmic laws were presented in a rather short theoretical part (12 out of 148 pages) entitled *On turbulent flow in pipes and along plates*. By and large, Prandtl arrived at the same results as Kármán. He duly acknowledged Kármán's publications from the year 1930, but he claimed that he had arrived at the same results at a time when Kármán's papers had not yet been known, so that once more, like ten years before with the same problem, the thoughts in Aachen and Göttingen followed parallel paths (Prandtl, 1932, p. 637). For the Hamburg conference proceedings, Prandtl and Eisner (1932) formulated a short appendix where they declared "that the priority for the formal [formelmässige] solution for the resistance of the smooth plate undoubtedly is due to Mr. v. Kármán who talked about it in August 1930 at the Stockholm Mechanics Congress."

When Kármán was finally aware of these publications, he felt embarrassed: "I hope that there will not remain an aftertaste from this debate," he wrote to Prandtl[34]. Prandtl admitted that he had "perhaps not without guilt" contributed to Kármán's misgivings. But he insisted[35] that his own version of the theory of plate resistance was better suited for practical use. Although the final results of Prandtl's and Kármán's approaches agreed with each other, there were differences with regard to the underlying assumptions and the ensuing derivations. Unlike Kármán, Prandtl did not start from a similarity hypothesis. There was no 'Kármán's constant' in Prandtl's version. Instead, when Prandtl accepted the logarithmic law as empirically given, he used the same dimensional considerations from which he had derived the 1/7th law from Blasius' empirical law. In retrospect, with the hindsight of Prandtl's notes[36], it is obvious that he came close to Kármán's reasoning – but the problem of how to account for the viscous range close to the wall (which Kármán bypassed by using the centerline of the channel as his vantage point) prevented a solution. In his textbook

[32] Kármán to Prandtl, 28 September 1932. MPGA, Abt. III, Rep. 61, Nr. 793.
[33] Prandtl to Kármán, 29 September 1932. MPGA, Abt. III, Rep. 61, Nr. 793.
[34] Kármán to Prandtl, 9 December 1932. MPGA, Abt. III, Rep. 61, Nr. 793.
[35] Prandtl to Karman, 19 December 1932. MPGA, Abt. III, Rep. 61, Nr. 793.
[36] Prandtl, notes, MPGA, Abt. III, Rep. 61, Nr. 2276, 2278.

Figure 2.4 Picture of the Rauhigkeitskanal at the Max Planck Institute for Dynamics and Self-Organization. It was built in 1935 and reconstituted by Helmut Eckelmann and James Wallace in the 1970s.

presentations (Prandtl, 1931, 1942a), he avoided the impression of a rivalry about the 'universal wall law' and duly acknowledged Kármán's priority.

But the rivalry between Prandtl and Kármán did not end with the conciliatory exchange of letters in December 1932. Kármán, who had moved in 1933 permanently to the USA, presented his own version of *Turbulence and Skin Friction* in the first issue of the new *Journal of the Aeronautical Sciences* (von Kármán, 1934). The Göttingen viewpoint was presented in textbooks such as Schlichting's *Boundary Layer Theory*, which emerged from wartime lectures that were translated after the war and first published as Technical Memoranda of NACA (Schlichting, 1949). The Göttingen school was also most active in elaborating the theory for practical applications which involved the consideration of pressure gradients (Gruschwitz, 1931) and roughness (Nikuradse, 1933; Prandtl, 1933E; Prandtl and Schlichting, 1934; Prandtl, 1934). After these basic studies, the turbulent boundary continued to be a major concern at Göttingen. By 1937–1938 the engineer Fritz Schultz-Grunow had joined the Institute and built (see Figure 2.4) a special 'Rauhigkeitskanal' – roughness channel – for the study of airplane surfaces (Schultz-Grunow, 1940), which was used subsequently for a variety of war-related turbulence research (Wieghardt, 1947; Prandtl, 1948a). This wind tunnel, which became the workhorse at KWI for measurements during 1939–45 (Wieghardt, 1942, 1943, 1944;

Wieghardt and Tillmann, 1944, 1951E) is the only tunnel that survived the dismantling at the end of the war as it was part of the KWI and not the AVA[37].

2.7 Fully developed turbulence I: 1932 to 1937

Prandtl's interest in fully developed turbulence[38] – beyond the quest for a 'universal wall law' – started in the early 1930s and is best captured following the regular correspondence he had with G.I. Taylor. Starting in 1923, Prandtl was regularly communicating with Taylor on topics in turbulence and instabilities. In 1923, after reading the seminal paper by Taylor (1923) on Taylor–Couette flow, Prandtl in his reply sent him a package of iron-glance powder (hematite) for flow visualization. It was the same material Prandtl had used for his visualization studies that led him to his 1904 discovery. He proposed[39] to Taylor to use it in his experiments, which Taylor immediately and successfully did. This initial contact led later to a very close relationship between the two giants of fluid mechanics. Although their relationship broke 15 years later on a disagreement about the politics of the Third Reich, their close relationship and the openness with which they communicated is impressive. Prandtl and Taylor were sometimes exchanging letters weekly. Prandtl visited Cambridge three times. The first time in 1927 was on the invitation by Taylor, the second time in 1934 for the Fourth International Congress of Applied Mechanics and the last time in 1936 to receive an honorary doctorate from Cambridge University. Prandtl would usually write[40] in typewritten German (in which Taylor's wife was fluent) while Taylor would reply in handwriting in English.

A letter from 1932 from Prandtl to Taylor marks a new stage with regard to turbulence. Prandtl refers to Taylor's recent work following the measurements of Fage and Falkner (see Chapter 4 by Sreenivasan):

> Your new theory of the wake behind a body and the experimental statement by Mr. Fage and Mr. Falkner on this matter reveals a very important new fact

[37] Private communication Helmut Eckelmann, and, as commented upon in the British Intelligence Objectives Sub-Committee report 760 that summarizes a visit at the KWI 26–30 April 1946, "Much of the equipment of the AVA has been or is in process of being shipped to the UK under MAP direction, but the present proposals for the future of the KWI Göttingen, appear to be that it shall be reconstituted as an institute for fundamental research in Germany under Allied control, in all branches of physics, not solely in fluid motion as hitherto. Scientific celebrities now at the KWI include Profs. Planck, Heisenberg, Hahn and Prandtl among others. In the view of this policy, it is only with difficulty that equipment can be removed from the KWI. The KWI records and library have already been reconstituted." Five months later the Max Planck Society was founded.

[38] It is important to note that all the research on turbulence in Göttingen was conducted at Prandtl's Kaiser Willhem Institute and not at the more technically oriented AVA.

[39] Prandtl to Taylor, 25 April 1923. MPGA, Abt. III, Rep. 61, Nr. 1653.

[40] Prandtl to Taylor, 5 June 1934. MPGA, Abt. III, Rep. 61, Nr. 1653.

> concerning turbulence. It demonstrates, that there are two different forms of turbulence, one belonging to the fluid motions along walls and the other belonging to mixture of free jets. In the first the principal axis of vorticity is parallel to the direction of the main flow, in the other this direction is perpendicular to the flow.

He continued that they found agreement with Taylor's theory at Göttingen by measuring the flow of a cold jet of air through a warm room. He closed the letter[41] with the following footnote:

> In the last weeks I studied your old papers from 1915 and 1922 with the greatest interest. I think, that if I had known these papers. I would have found the way to turbulence earlier.

Thus Prandtl concluded that there are two kinds of turbulence, one being wall-turbulence and the other jet-turbulence (Prandtl, 1933). A third kind of fully developed turbulence, the turbulence in a wind tunnel, had appeared on Prandtl's agenda as early as in 1921, when Hugh Dryden from the National Bureau of Standards in Washington, DC, had asked[42] him about "a proper method of defining numerically the turbulence of tunnels and your idea as to the physical conception of the turbulence". Already then Prandtl was considered the pioneer in wind tunnel design as is reflected in his instructions that he wrote in 1932 in the *Handbook of Experimental Physics* and that were translated shortly thereafter into English (Prandtl, 1933E2).

In his response[43] Prandtl had pointed to the vortices in the turbulent air stream that

> are carried with the flow and are in time consumed by the viscosity of the air. In a turbulent flow the velocity of the flow is changing in space and time. Characteristic quantities are the average angular velocity of the vortex and the diameter, whereby one has to think of a statistical distribution, in which vortices of different sizes and intensity coexist next to each other.

However, without the appropriate means for measuring these quantities, the problem disappeared again from his agenda – until the 1930s, when the isotropic turbulence behind a grid in a wind tunnel was measured with sophisticated new techniques. Fage and Townend (1932) (see also Collar, 1978) had investigated the full three-dimensional mean flow and the associated three-dimensional average velocity fluctuations in turbulent channel and pipe flow using particle tracking streak images of micron size tracers with a microscope. In addition, Dryden and Kuethe (1929) (see also Kuethe, 1988) invented compensated hotwire measurements, which were to revolutionize the field of

[41] Prandtl to Taylor, 25 July 1932. MPGA, Abt. III, Rep. 61, Nr. 1653.
[42] Dryden to Prandtl, 6 March 1921. MPGA, Abt. III, Rep. 61, Nr. 362.
[43] Prandtl to Dryden, 20 April 1921. MPGA, Abt. III, Rep. 61, Nr. 362.

turbulence measurements. This set the stage for spectral measurements of turbulent velocity fluctuations.

From two letters between Taylor and Prandtl in August and December 1932, following the correspondence[44] discussed above, it is apparent that both had started to conduct hotwire experiments to investigate the turbulent velocity fluctuations: Taylor in collaboration with researchers at the National Physical Laboratory (NPL) – most likely Simmons and Salter (see Simmons et al., 1938) – and Prandtl with Reichardt (see Reichardt, 1933; Prandtl and Reichardt, 1934). Taylor responded[45] with suggestions for pressure correlation measurements and argued:

> The same kind of analysis can be applied to hotwire measurements and I am hoping to begin some work on those lines. In particular the 'spectrum of turbulence' has not received much attention.

Prandtl replied[46]:

> I do not believe that one can achieve a clear result with pressure measurements, as there is no instrument that can measure these small pressure fluctuations with sufficient speed. Instead hotwire measurements should lead to good results. We ourselves have conducted an experiment in which two hotwires are placed at larger or smaller distances from each other and are, with an amplifier, connected to a cathode ray tube such that the fluctuations of the one hotwire appear as horizontal paths, and those of the other as perpendicular paths on the fluorescent screen[47] ... To measure also the magnitude of the correlation my collaborator Dr. Reichardt built an electrodynamometer with which he can observe the mean of u'_1, u'_2 and $u'_1 u'_2$. In any case, I am as convinced as you that from the study of those correlations as well as between the direction and magnitude fluctuations, for which we have prepared a hotwire setup, very important insights into turbulent flows can be gained.

In the same letter Prandtl sketched three pictures of the deflections of the oscilloscope that are also published in Prandtl and Reichardt (1934). In this article the authors reported that the hotwire measurements leading to the figures had been finished in August 1932 (date of the letter of Taylor to Prandtl), and that in October 1933 a micro-pressure manometer had been developed to measure the very weak turbulent fluctuations.[48] It is very remarkable that it took less than a year for Prandtl and Reichardt to pick up the pressure measurement proposal by Taylor. It also shows the technical ingenuity and the excellent mechanics workshop at the Göttingen KWI. The micro-pressure

[44] Prandtl to Taylor, 25 July 1932. MPGA, Abt. III, Rep. 61, Nr. 1653.
[45] Taylor to Prandtl, 18 August 1932. MPGA, Abt. III, Rep. 61, Nr. 1653.
[46] Prandtl to Taylor, 23 December 1932. MPGA, Abt. III, Rep. 61, Nr. 1653.
[47] This way of showing correlations had been used at the KWI since 1930 (Reichardt, 1938b).
[48] See the paper Reichardt (1934) which was at that time in preparation.

gauge first described in Reichardt (1935, 1948E) is still a very useful design.

This exchange of letters marks the beginning of the correlation and spectral analysis of turbulent fluctuations that are at the foundation of turbulence research even today. Only three weeks later, Taylor replied[49] from a skiing vacation in Switzerland. He relied on the NPL with regard to wind tunnel measurements. They measured the spectrum of turbulence behind a screen of equally spaced rods and found it to settle down to a time error function for which he had no theoretical explanation. Prandtl suggested in his reply that the frequency spectrum behind a grid made of rods may be attributable to von Kàrmàn vortices. He added that "apart from this one needs to wait for the publication". Finally he asked[50] whether Taylor could have his letters rewritten by someone else in more legible writing as he had problems in deciphering Taylor's handwriting. This seems to have caused an interruption of their communication on turbulence for a while.

The next letter[51] in the MPG-Archive is from June 1934. Taylor invited Prandtl to stay in his house during the upcoming Fourth International Congress for Applied Mechanics, to be held in Cambridge during 3–9 July 1934. Prandtl answered[52] in a quite formal and apologetic manner: "In reply to your exceedingly friendly lines from 1.6.34 I may reply to you in German, as I know that your wife understands German without difficulty."

The discussion on turbulence came back to full swing after Prandtl's 60th birthday on 4 February 1935, with almost weekly correspondence. The year 1935 was the one in which Taylor published what many regard as his most important papers in turbulence (Taylor, 1935a,b,d,e). The correspondence between the two that year seems to have greatly influenced those papers. Taylor contributed as the only non-German scientist to the Festschrift published in *ZAMM* and handed to Prandtl at the occasion of his birthday. In his article Taylor compared his calculation of the development of turbulence in a contraction with independent measurements by Salter using a hot wire, as well as photographs by Townend of spots of air heated by a spark and by Fage using his ultramicroscope (Taylor, 1935a). In other words, the best English fluid-dynamicists contributed to this Festschrift.

Prandtl thanked Taylor immediately[53] asking him about details of the paper. The reply from Taylor convinced Prandtl of the correctness of Taylor's work. As a sideline, Taylor also mentioned that turbulence after a constriction

49 Taylor to Prandtl, 14 January 1933. MPGA, Abt. III, Rep. 61, Nr. 1653.
50 Prandtl to Taylor, 25 January 1933. MPGA, Abt. III, Rep. 61, Nr. 1653.
51 Taylor to Prandtl, 1 June 1934. MPGA, Abt. III, Rep. 61, Nr. 1653.
52 Prandtl to Taylor, 5 June 1934. MPGA, Abt. III, Rep. 61, Nr. 1653.
53 Prandtl to Taylor, 28 February 1935. MPGA, Abt. III, Rep. 61, Nr. 1654.

readjusts itself into a condition where the turbulent velocities are much more nearly equally distributed in space.

(This was later investigated in detail in Comte-Bellot and Corrsin, 1966.) In the same letter[54] Taylor informed Prandtl that

> I lately have been doing a great deal of work on turbulence ... In the course of my work I have brought out two formulae which seem to have practical interest. The first concerns the rate of decay of energy in a windstream ... and I have compared them with some of Dryden's measurements behind a honeycomb – it seems to fit. It also fits Simmons' measurements with turbulence made on a very different scale ...

The second formula was concerned with the "theory of the critical Reynolds number of a sphere behind a turbulence-grid", as Prandtl replied in his letter[55] pointing him to his own experimental work from 1914. Prandtl also mentioned that measurement from Göttingen see a signature of the grid. Taylor interpreted this as the "shadow of a screen", which according to Dryden's experiments dies away after a point, where the turbulence is still fully developed.[56] It was this region where Taylor expected his theory to apply. Taylor submitted his results in four consecutive papers "On the statistics of turbulence" on 4 July 1935 (Taylor, 1935a, 1935b, 1935d, 1935e). Later Prandtl re-derived Taylor's decay law of turbulence (Wieghardt, 1941, 1942E; Prandtl and Wieghardt, 1945). A detailed discussion of the physics of the decay law of grid-generated turbulence can be found in Chapter 4 by Sreenivasan.

A month later Taylor[57] thanked Prandtl for sending him his paper with Reichardt (Prandtl and Reichardt, 1934) on measurements of the correlations of turbulent velocity fluctuations that Prandtl had already referred to in his letter[58] in 1932. Taylor needed these data for "comparison with my theory of energy dissipation".

Again the correspondence with Prandtl surely contributed to Taylor's understanding and finally led to the third paper in the 1935 sequence (Taylor, 1935d). After returning from the 5th Volta Congress in Rome (on high speeds in aviation), which both attended, Prandtl mentioned[59] to Taylor that "Mr. Reichardt conducts new correlation measurements this time correlations between u' and v'. The results we will send in the future". Again, just as in 1933[60],

[54] Taylor to Prandtl, 2 March 1935. MPGA, Abt. III, Rep. 61, Nr. 1654.
[55] Prandtl to Taylor, 12 March 1935. MPGA, Abt. III, Rep. 61, Nr. 1654.
[56] Taylor to Prandtl, 14 March 1935. MPGA, Abt. III, Rep. 61, Nr. 1654.
[57] Taylor to Prandtl, 21 April 1935. MPGA, Abt. III, Rep. 61, Nr. 1654.
[58] Prandtl to Taylor, 23 December 1932. MPGA, Abt. III, Rep. 61, Nr. 1653.
[59] Prandtl to Taylor, 12 November 1935. MPGA, Abt. III, Rep. 61, Nr. 1654.
[60] Prandtl to Taylor, 25 January 1933. MPGA, Abt. III, Rep. 61, Nr. 1653.

where Prandtl used a similar formulation, the correspondence does not return to the matter of turbulence until more than a year later.

They resumed the discussion again when Taylor sent Prandtl a copy of his paper on "Correlation measurements in a turbulent flow through a pipe" (Taylor, 1936). Prandtl responded[61] that "currently we are most interested in the measurement of correlations between locations in the pipe", suggesting that Taylor may consider measurements away from the center of the pipe and mentioned that "the measurements in Fig. 4 agree qualitatively well with our u' measurements. A better agreement is not to be expected as we measured in a rectangular channel and you in a round pipe." Taylor replied on 11 January 1937 and also on[62] 23 January 1937 when he sent Prandtl "our best measurements so that you may compare with your measurements in a flat pipe".

Thus by 1937 the stage was set at Göttingen and Cambridge for the most important measurements about the statistics of turbulent fluctuations. At the same time, Dryden and his co-workers at the National Bureau of Standards in Washington measured the decay of the longitudinal correlations behind grids of different mesh sizes M (Dryden et al., 1937) and calculated from it by integration of the correlation function the integral scale of the flow, what Taylor (1938b, p. 296) called the "the scale of turbulence". By using different grids they were able to show that the grid mesh size M determined the large-scale L of the flow, just as Taylor had assumed in 1935. They also found that the relative integral scale L/M increased with the relative distance x/M from the grid, independently of M. This was later analyzed in more detail by Taylor (1938b) with data from the National Physical Laboratory in Teddington. The paper by Dryden and collaborators (Dryden et al., 1937) was very important for the further development of turbulence research, as it was data from here that Kolmogorov used in 1941 for comparison with his theory (see Chapter 6 on the Russian school by Falkovich).

2.8 Fully developed turbulence II: 1938

After Taylor's paper on the "Spectrum of turbulence" appeared (Taylor, 1938a), Taylor answered a previous letter[63] by Prandtl. He first thanked Prandtl for sending him Reichardt's $u'v'$ correlation data in a channel flow (Reichardt, 1938a,b; 1951E), which he regarded as "certainly of the same type" as those

[61] Prandtl to Taylor, 9 January 1937. MPGA, Abt. III, Rep. 61, Nr. 1654.
[62] Taylor to Prandtl, 23 January 1937. MPGA, Abt. III, Rep. 61, Nr. 1654.
[63] Taylor to Prandtl, 18 March 1938. MPGA, Abt. III, Rep. 61, Nr. 1654. Prandtl's letter which prompted this response has not yet been found.

of Simmons for the round pipe. Then he answered a question of Prandtl about the recent paper on the spectrum of velocity fluctuations (Taylor, 1938a) and explained to him what we now know as 'Taylor's frozen flow hypothesis', i.e.

> that the formula depends only on the assumption that u is small compared to U so that the succession of events at a point fixed in the turbulent stream is assumed to be related directly to the Fourier analysis of the (u, x) curve obtained from simultaneous measurements of u and x along a line parallel to the direction of U.

It is interesting to note that Taylor had a clear concept of the self-similarity of grid-generated turbulence:

> The fact that increasing the speed of turbulent motion leaves the curve $\{UF(n), n/U\}$ unchanged except at the highest levels of n means that an increase in the 'Reynolds number of turbulence' leaves the turbulence pattern unchanged in all its features except in the components of the highest frequency.

Six months after this exchange, Prandtl and Taylor met in Cambridge, MA, for the Fifth International Congress for Applied Mechanics, held at Harvard University and the Massachusetts Institute of Technology during 12–16 September 1938. By now the foundations for seminal discoveries in turbulence research were set. For the next 60 years, experimental turbulence research was dominated by the Eulerian approach, i.e. spatial and equal time measurements of turbulent fluctuations as introduced in the period 1932–1938.

At this Fifth International Congress turbulence was the most important topic.

> In the view of the great interest in the problem of turbulence at the Fourth Congress and of the important changes in accepted views since 1934 it was decided to hold a Turbulence Symposium at the Fifth Congress. Professor Prandtl kindly consented to organize this Symposium and ... The Organizing committee is grateful to Professor Prandtl and considers his Turbulence Symposium not only the principal feature of this Congress, but perhaps the Congress activity that will materially affect the orientation of future research

wrote Hunsaker and von Kármán in the 'Report of the Secretaries' in the conference proceedings. Prandtl had gathered the leading turbulence researchers of his time to this event. The most important talks, other than the one by Prandtl, were the overview lecture by Taylor on "Some recent developments in the study of turbulence" (Taylor, 1938b) and Dryden's presentation with his measurements of the energy spectrum (Dryden, 1938). It is interesting to note that, in concluding his paper, Dryden presented a single hotwire technique that he intended to use to measure the turbulent shearing stress $u'v'$. Prandtl's laboratory under Reichardt's leadership had already found a solution earlier[64]

[64] Taylor to Prandtl, 18 March 1938. MPGA, Abt. III, Rep. 61, Nr. 1654.

and Prandtl presented this data in his talk. Reichardt used hotwire anemometry with a probe consisting of three parallel wires, where the center wire was mounted a few millimeters downstream and used as a temperature probe (Reichardt, 1938a,b; 1951E). The transverse component of velocity was sensed by the wake of one of the front wires. This probe was calibrated in oscillating laminar flow. During the discussion of Dryden's paper, Prandtl made the following important remark concerning the turbulent boundary layer: "One can assume that the boundary zone represents the true 'eddy factory' and the spread towards the middle would be more passive." In his comment, he also showed a copy of the fluctuation measurements conducted by Reichardt and Motzfeld (Reichardt, 1938b, Fig. 3) in a wind tunnel of 1 m width and 24 cm height.

Prandtl's 1938 paper at the turbulence symposium deserves a closer review because, on the one hand, it became the foundation for his further work on turbulence and, on the other, as an illustration of Prandtl's style. It reflects beautifully and exemplarily, what von Kármán, for example, admired as Prandtl's "ability to establish systems of simplified equations which expressed the essential physical relations and dropped the nonessentials"; von Kármán regarded this ability "unique" and even compared Prandtl in this regard with "his great predecessors in the field of mechanics – men like Leonhard Euler (1707–1783) and d'Alembert (1717–1783)".[65]

In this paper, Prandtl distinguished four types of turbulence: wall turbulence, free turbulence, turbulence in stratified flows (see also Prandtl and Reichardt, 1934), and the decaying isotropic turbulence. He first considered the decay law of turbulence behind a grid using his mixing length approach by assuming that the fluctuating velocity is generated at a time t' and then decays:

$$u'^2 = \int_{-\infty}^{t} \frac{\mathrm{d}t'}{T} \left[\left(l_1 \frac{dU}{dy} \right)_{t'} \times f\left(\frac{t-t'}{T} \right) \right]^2,$$

with $f(\frac{t-t'}{T}) \approx \frac{T}{T+t-t'}$ justified by Dryden's measurements. With the shear stress $\tau = \rho u' l_2 \frac{dU}{dy}$ and $l_2 = km$, where m is the grid spacing and $k \approx 0.103$, he derived

$$u' = \frac{\text{const}}{t+T} = \frac{cU_m}{x},$$

where x is the downstream distance from the grid, c is related to the thickness of the rods, and U_m is the mean flow. From the equation of motion to lowest order,

$$U_m \frac{\partial U}{\partial x} = \frac{1}{\rho} \frac{\partial \tau}{\partial y},$$

[65] See Kármán (1957); Anderson (2005); see also Prandtl's own enlightening contribution to this topic (Prandtl, 1948b).

and the Ansatz

$$U = U_m + Ax^{-n} \cos\left(\frac{2\pi y}{m}\right),$$

he obtained $n = 4\pi^2 k \frac{c}{m}$. By analyzing the largest frequency component of $U - U_m$ (from data provided by Dryden) he determined $n \approx 4.5$. This result led him to assume a transition from anisotropic flow near the grid to isotropic turbulence further downstream. How this transition occurs was left open.

As a next item, Prandtl considered the change of a wall-bounded, turbulent flow at the transition from a smooth to a rough wall and vice versa. He derived model equations and found reasonable agreement with the measurements of his student Willi Jacobs.

The third item of Prandtl's conference paper concerned an ingeniously simple experiment. By visualizing the flow with iron-glance flakes (the same as he had used in his 1904 experiments) he measured what he perceived as the 'Taylor scale' of turbulence in a grid-generated turbulent water flow (see Figure 2.5; in the region of large shear the flakes align and make visible the eddies in the turbulent flow). From the surface area per eddy as a function of mesh distances behind the grid (see Prandtl, 1938, Fig. 1) he found that these areas grew linearly starting from about 16 mesh distances downstream from the grid. From this observation Prandtl concluded that the Taylor scale λ (Taylor, 1935b) increases as $(x - x_0)^{0.5}$, where $x_0 \approx 10$ is the 'starting length'. This was in contradiction with Taylor's own result, but agreed with the prediction by Kármán and Howarth (1938) from eight months earlier. (It is not clear whether Prandtl knew about their work – it is not credited in his paper. Later the von Kármán and Howarth prediction was quantitatively measured with hot wires by Comte-Bellot and Corrsin, 1966.)

Finally, and in retrospect most importantly, Prandtl discussed Reichardt's and Motzfeld's measurements of wall-generated turbulence in channel flow (Reichardt, 1938a,b; 1951E; Motzfeld, 1938). In Figure 2.6(a) we reproduce his Fig. 2. It displays the mean fluctuating quantities $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$, $\overline{v'u'}$, and $\Psi = \overline{v'u'}/(\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}})$ as a function of distance from the wall to the middle of the tunnel at 12 cm. Here u is the streamwise, v the wall normal, and w wall parallel velocity.

The spectral analysis of the streamwise velocity fluctuation u as a function of frequency revealed that the frequency power spectra were indistinguishable for distances of 1 to 12 cm from the wall, although $\sqrt{\overline{u'^2}}$ decreased by more than a factor of 2 over the same range (see Figure 2.6(a), reproduced from Motzfeld, 1938). Prandtl showed in his Fig. 4 (here Figure 2.6(b)) the semi-log plot of $nf(n)$, which displays a maximum at the largest scales of

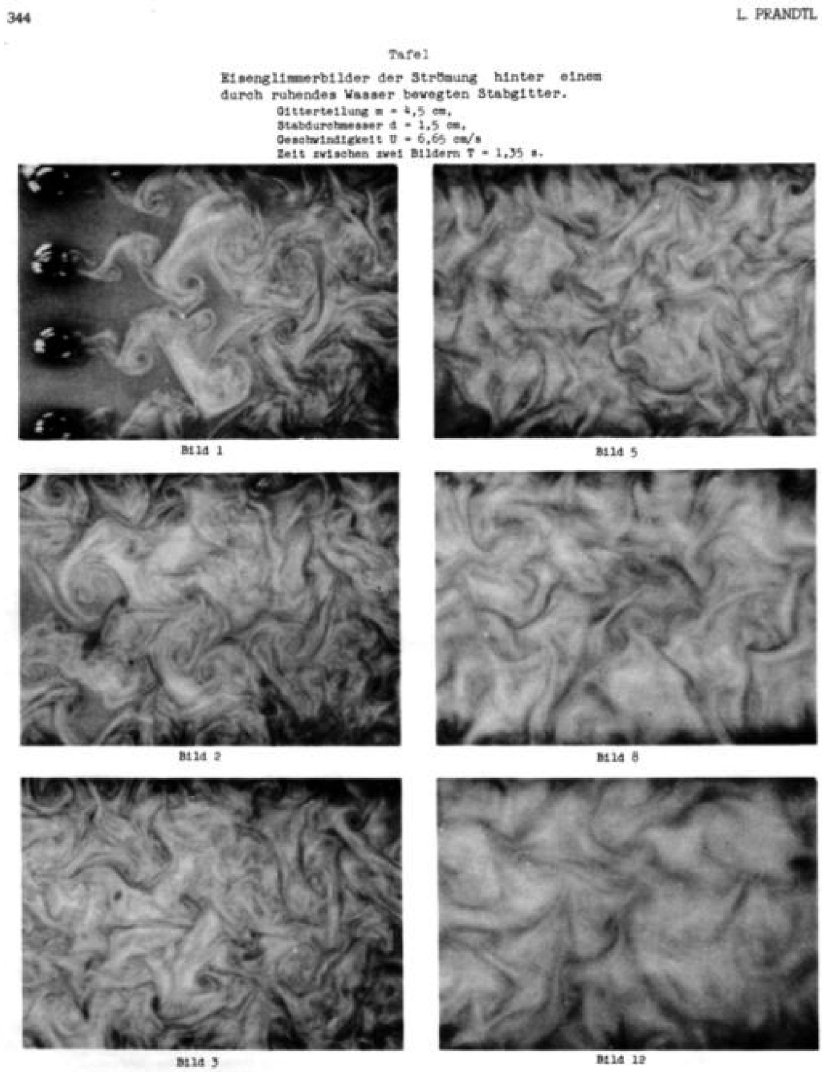
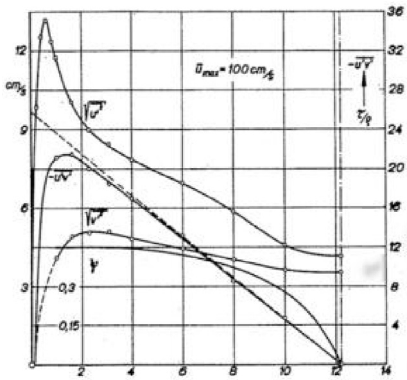
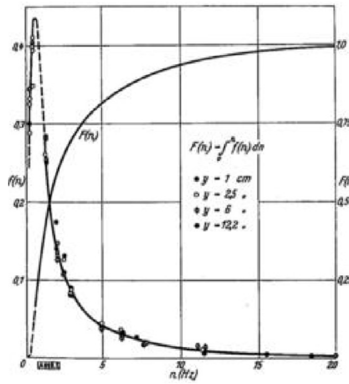


Figure 2.5 Prandtl's visualization of the development of grid-generated isotropic turbulence. Pictures were taken at relative grid spacings of 2, 4, 6, 10, 16 and 24.

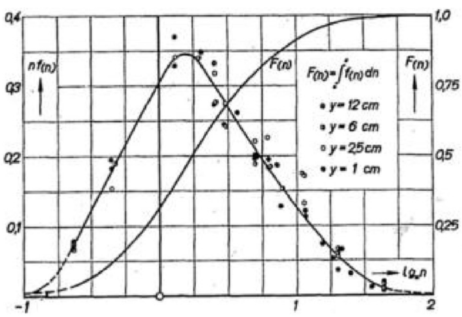
the flow. Later (Lumley and Panofsky, 1964), the location of this maximum was proposed as a surrogate for the integral scale. As shown in Figure 2.6(c), Motzfeld also compared his data with the 1938 wind tunnel data in Simmons et al. (1938) by rescaling both datasets with the mean velocity U and the



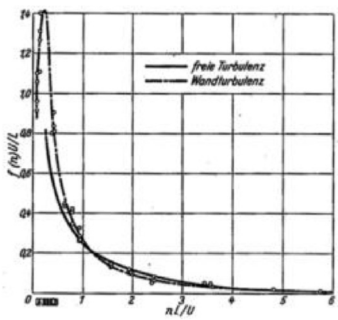
(a) Fluctuating quantities measured in a flow between two parallel plates for Re_{max} = 17500.



(b) Velocity spectra for different distances from the wall in turbulent channel flow (Wandturbulenz).



(c) Log/lin plot of the spectrum in (a)



(d) Velocity spectrum compared between shear-generated and grid-generated decaying turbulence

Figure 2.6 Turbulence spectra as measured by Motzfeld and Reichardt in 1938.

channel height L or, for the wind tunnel, with the grid spacing L. As we can see both datasets agree reasonably well. Prandtl remarked about the surprising collapse of the data in Figure 2.6(a) (Fig. 3 in Prandtl, 1938): “The most remarkable about these measurements is that *de facto* the same frequency distribution was found.” From the perspective of experimental techniques, the electromechanical measurement technique employed by Motzfeld and Reichardt is also remarkable. As shown in Figure 2.7, they used the amplitude of an electromechanically driven and viscously damped torsion wire resonant oscillator to measure, by tuning resonance frequencies and damping, the frequency components of the hotwire signal.

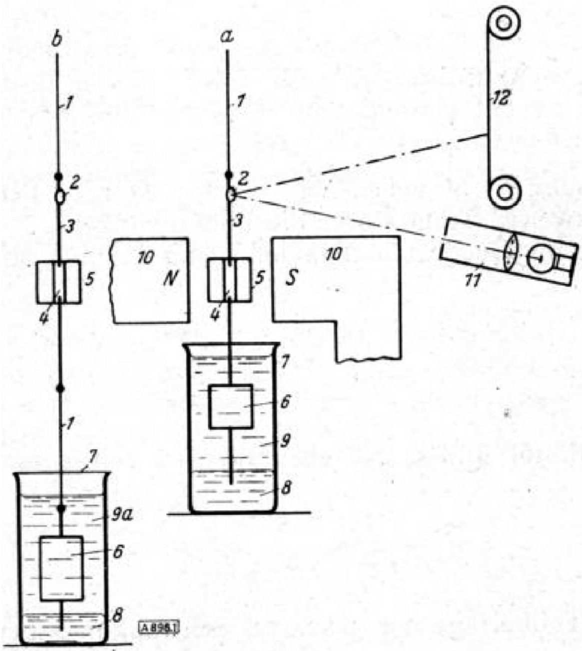


Figure 2.7 Schematic of the electromechanical spectral analysis system used by Motzfeld in 1937/38. Two different designs of a damped torsion pendulum were used: (a) below 20 Hz and (b) above 20 Hz. The design (a) consisted of a torsion wire (1), a thin metal rod (3) with mirror (2), an insulating glass rod (4) around which a coil was wound (5), a swinger consisting of a thin metal rod with a cylindrical body to add inertia (6). The swinger was placed in a beaker (7) that was filled on the top with a damping fluid (9) and on the bottom with mercury (8). Electric currents could flow from (1) into the coil and from there to (8). The electromagnet was placed in a permanent magnetic field. The deflections of the wire were recorded on photographic film that was transported with a motor. In the alternative design (b) for more than 20 Hz the swinger was replaced with a torsion wire (1) and a weight (6). The weight was placed into very viscous oils so that it did not move (9a). Otherwise the design was the same. Altogether ten swingers were used with six of kind (a) and four of kind (b). The resonance frequencies were between 0.2 Hz and 43 Hz.

As we will see, these results would lead (Prandtl and Wieghardt, 1945) not only to the derivation of what is now known as the ‘one equation model’ (Spalding, 1991), but also to the assumption of a universal energy cascade of turbulence cut off at the dissipation scale (Prandtl, 1945, 1948a). Thus, at age 70, Prandtl had finally found what he had been looking for all his life albeit at the worst time – when the Second World War ended and he was not allowed to conduct scientific research.[66]

Prandtl's sojourn in the USA in September 1938 was also remarkable in another respect – because it marked the beginning of his, and for that matter Germany's, alienation from the international community. When he tried to convince the conference committee to have him organize the next Congress in Germany, he encountered strong opposition based on political and humanitarian reasons. Against many of his foreign colleagues, Prandtl defended Hitler's politics and actions. Taylor attempted to cure Prandtl of his political views.[67] As discussed also in Chapter 4 by Sreenivasan, Taylor was concerned with the humanitarian situation of the Jewish population and the political situation in general. However, Taylor's candor (he called Hitler "a criminal lunatic") did not bode well with Prandtl, who responded again by defending German politics.[68] Only a few days before Prandtl wrote his letter (on 18 October 1938) 12,000 Polish-born Jews were expelled from Germany. On 11 November 1938, the atrocities of the 'Kristallnacht' started the genocide and Holocaust (Gilbert, 2006). Taylor replied on 16 November 1938 with a report about the very bad experiences in Germany of his own family members.[69] Nevertheless he ended his letter still quite amicably:

> You will see that we are not likely to agree on political matters so it would be best to say no more about them. Fortunately there is no reason why people who do not agree politically should not be best friends.

Then he continued to make a remark that he does not understand why Prandtl plotted $nf(n)$ instead of $f(n)$ (shown in Figure 5b) (Fig. 4 in Prandtl, 1938). As far as we know Prandtl never replied. After this correspondence Prandtl wrote one more letter to Mrs Taylor.[70] Only a month later World War II started and cut off their communication. Prandtl tried[71] to resume contact with Taylor after the war, but there is no evidence that Taylor ever responded to this effort.

2.9 Fully developed turbulence III: 1939 to 1945

With the beginning of WWII on 1 September 1939, German research in fluid dynamics became isolated from the rest of the world. This may explain why

[66] Prandtl to Taylor, 18 July 1945 and 11 October 1945. MPGA, Abt. III, Rep. 61, Nr. 1654
[67] Taylor to Prandtl, 27 September 1938. MPGA, Abt. III, Rep. 61, Nr. 1654.
[68] Prandtl to Taylor, 29 October 1938. MPGA, Abt. III, Rep. 61, Nr. 1654.
[69] Taylor to Prandtl, 16 November 1938. GOAR 3670-1
[70] Prandtl to Mrs Taylor, 5 August 1939. MPGA, Abt. III, Rep. 61, Nr. 1654.
[71] Prandtl to Taylor, 18 July 1945 and 11 October 1945. MPGA, Abt. III, Rep. 61, Nr. 1654

the very important discovery by Motzfeld and Reichardt was not recognized abroad. We have found no reference to Motzfeld's 1938 publication other than in the unpublished 1945 paper by Prandtl (see below). As described above their discovery showed that the spectrum of the streamwise velocity fluctuation in a channel flow did not depend on the location of the measurements in the channel and did agree qualitatively with those by Simmons and Salter for decaying isotropic turbulence. The 1938 Göttingen results show beautifully the universal behavior that Kolmogorov postulated in his revolutionary 1941 work (see Chapter 6 on the Russian school by Falkovich).

Prandtl and his co-workers were not aware of the developments in Russia and continued their program in turbulence at a slower pace. According to a British Intelligence report[72] after the war, based on an interrogation of Prandtl,

> due to more urgent practical problems little fundamental work, either experimental or theoretical, had been conducted during the war. No work had been done in Germany similar to that of G.I. Taylor or Kármán and Howarth on the statistical theory of turbulence. Experiments had been planned on the decay of turbulence behind grids in a wind tunnel analogous to those undertaken by Simmons at the National Physical Laboratories, but these were shelved at the outbreak of the war.

Indeed as far as fully developed turbulence was concerned the progress was mostly theoretical and mainly relying on measurements made before the war. In his response to the military interrogators, Prandtl was very modest. From late autumn of 1944 till the middle of 1945 he worked on the theory of fully developed turbulence almost daily (see Figure 2.8). This was his most active period in which he pulled together the threads outlined earlier.

We will now review briefly the development from 1939 to 1944 that led to this stage. The status of knowledge of turbulence in 1941 is well summarized in Wieghardt (1941; 1942E), and that between 1941 and 1944 in Prandtl's (1948a) FIAT article entitled *Turbulenz*. Prandtl reviewed in 23 tightly written pages the work at the KWI in chronological order and by these topics:

- (i) Turbulence in the presence of walls
  - (a) Pipeflow
  - (b) Flat plates
  - (c) Flow along walls with pressure increase and decrease
- (ii) Free turbulence
  - (a) General laws

[72] British Intelligence Objectives Sub-Committee report 760 that summarizes a visit to the KWI, 26–30 April 1946.

October 1944

Mo	Tu	We	Th	Fr	Sa	Su
						1
2	3	4	5	6	7	8
9	10	11	12	13	T1	T1
T1	17	T1	19	20	T1	22
23	24	25	26	27	28	29
30	"E					

November 1944

Mo	Tu	We	Th	Fr	Sa	Su
		1	T2	3	T2	5
T2	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	T1
T1	T1	29	30			

December 1944

Mo	Tu	We	Th	Fr	Sa	Su
				1	2	T1
4	T1	T1	T1	8	9	T1
11	12	T2	14	15	16	T2
T2	19	T2	T1	T1	23	24
25	T2	27	28	29	30	31

January 1945

Mo	Tu	We	Th	Fr	Sa	Su
T1	2	T1	S	5	6	T1
8	9	10	11	12	13	14
15	16	17	18	19	20	T2
22	23	24	25	A	27	28
K41	30	31				

February 1945

Mo	Tu	We	Th	Fr	Sa	Su
			1	2	3	B
5	6	7	9	T2	10	C
12	13	14	T2	T2	17	18
19	20	21	22	23	T2	25
26	27	28				

March 1945

Mo	Tu	We	Th	Fr	Sa	Su
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	R2	T2	T2	30	31	

April 1945

Mo	Tu	We	Th	Fr	Sa	Su
						1
2	3	4	5	6	7	O
9	10	11	12	13	14	15
16	T2	T2	19	20	21	22
23	24	25	26	25	28	29
30						

May 1945

Mo	Tu	We	Th	Fr	Sa	Su
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

June 1945

Mo	Tu	We	Th	Fr	Sa	Su
				1	2	3
4	5	6	7	8	9	T2
11	12	T2	14	15	T2	17
18	19	20	21	22	23	T2
25	26	27	28	29	30	

July 1945

Mo	Tu	We	Th	Fr	Sa	Su
						1
2	3	*DP*	5	6	7	8
9	10	11	12	13	14	V
V	17	18	19	V	21	22
23	24	25	26	27	28	TS
TS	31					

August 1945

Mo	Tu	We	Th	Fr	Sa	Su
		1	TS	3	4	5
6	7	8	9	10	11	TS

Figure 2.8 Prandtl worked continuously on the topic of fully developed turbulence. T1 marks Prandtl's work on the energy equation of turbulence, T2 his investigations on the effect of dissipation, V a derivation of the vorticity equation in a plane shear flow, and TS his attempts to develop a statistical theory of velocity fluctuations. The other letters mark important dates: on 31 October 1944 he formulated for the first time the 'one equation model' (E); on 4 January 1945 he presented it at a theory seminar (S) and on 26 January 1945 at a meeting of the Göttingen Academy of Science (A); 29 January 1945 marks his discovery of what is known as the Kolmogorov length scale (K41); on 4 February 1945 he had his 70th birthday (B); on 11 February 1945 he formulated for the first time his cascade model (C); on 27 March 1945 he is reworking the draft for the paper on dissipation (R2); on 8 April 1945 Göttingen was occupied by American forces; on 4 July 1945 Prandtl entered remarks on the already typewritten draft revisions of the dissipation paper. The period in May, where he had no access to the Institute as it was used by American forces, is light gray – the Institute reopened on 4 June 1945 to close again briefly thereafter.

- (b) Special tasks
- (c) Properties of jets in jet engines

(iii) Various investigations

- (a) Turbulence measurement technologies
- (b) Heat exchange
- (c) Geophysical applications
- (d) Fundamental questions

Prandtl identified as fundamental and important in particular the work by Schultz-Grunow (1940; 1941E) and Wieghardt (1944) on the measurements of the turbulent boundary layer. Even today, these very careful and now classical experiments provide the data for quantitative comparisons (Nagib et al., 2007).

Furthermore, Prandtl singled out the investigations on heat transfer in turbulent boundary layers by Reichardt (1944), who applied ideas from earlier papers on turbulent transport of momentum in a free jet (Reichardt, 1941, 1942). Reichardt had found experimentally for a planar jet that the PDF of transverse variations of the streamwise velocity profile in the middle of a jet was Gaussian. In the middle of such flows $\frac{\partial \bar{u}}{\partial y} = 0$, where the mixing length approach failed by design, as Prandtl (1925) had already noted, when he suggested another way around this problem. Based on the observation of the Gaussian distribution he conjectured inductively that the transfer of momentum was similar to that of heat. By neglecting viscosity he wrote down the two-dimensional planar momentum equation

$$\frac{\partial}{\partial x}(p/\rho + \overline{u^2}) + \frac{\partial \overline{(uv)}}{\partial y} = 0$$

and

$$\overline{uv} = -\lambda \frac{\partial \overline{u^2}}{\partial y},$$

with λ as *Übertragunsgrösse* (transfer quantity). Reichardt calculated some examples and showed that the new theory worked reasonably well. Prandtl (1942c) had already published about it in *ZAMM*. He showed that if the pressure term in lowest order is zero the two equations by Reichardt lead to

$$\frac{\partial}{\partial x}\overline{u^2} = -\lambda \frac{\partial^2 \overline{u^2}}{\partial y^2}.$$

In a subsequent paper Henry Görtler (1942) applied the theory to four cases: the plane mixing layer, the plane jet, the plane wake and the plane grid. He compared the first two cases with the measurements by Reichardt and found good agreement.

As another important result Prandtl highlighted improvements of the hotwire measurement system by H. Schuh who had found a method for circumventing the otherwise very tedious calibration of each new hotwire probe in a calibration tunnel (Schuh, 1945, 1946). With regard to theoretical achievements, Prandtl reported on the work of the mathematician Georg Hamel who had proved von Kármán's 1930 similarity hypothesis for the two-dimensional flow in a channel as well as Prandtl's log law (Hamel, 1943; Prandtl, 1925).

At the end of the FIAT paper, Prandtl mentioned rather briefly what he had been so deeply engaged in from the autumn of 1944 to the summer of 1945. In only a little more than a page he summarized his energy model of turbulence (the 'one equation model'; Spalding, 1991) and his own derivation of the Kolmogorov length scales, for which he used a cascade model of energy transfer to the smallest scales. The latter he attributed to his unpublished manuscript from 1945 (see the discussion below). Then he reviewed the work by Weizsäcker (1948) and Heisenberg (1948; 1958E) that they had conducted while detained in England from July 1945 to January 1946.[73] Weizsäcker's work was similar, but Prandtl considered it to be mathematically more rigorous than his phenomenologically driven approach. In addition to the results that Prandtl had obtained, Weizsäcker also calculated from the energy transport the $k^{-5/3}$ scaling of the energy spectrum. Prandtl then reviewed the Fourier mode analysis of Heisenberg and commented on the good agreement with experiments. He closed with a hint at a paper in preparation by Weizsäcker concerning the influence of turbulence on cosmogony.

2.10 Prandtl's two manuscripts on turbulence, 1944–1945

When the American troops occupied Göttingen on 8 April 1945, Prandtl had already published the 'one equation model' (Prandtl and Wieghardt, 1945) and drafted a first typewritten manuscript of a paper entitled 'The role of viscosity in the mechanism of developed turbulence' that was last dated by him 4 July 1945 (Prandtl, 1945; see Figure 2.8). In this paper he derived from a cascade model the dissipation length scale, i.e. the 'Kolmogorov length'. Before we describe in more detail his discoveries, it is important to ask why he did not publish this work. Clearly this was an important discovery and would have retrospectively placed him next to Kolmogorov in the "remarkable series of coincidences" (Batchelor, 1946, p. 883) now known as the K41 theory.

[73] Their work had also been reviewed by Batchelor in December 1946 together with the work of Kolmogorov and Onsager (Batchelor, 1946). Of course Batchelor had no knowledge of the fact that Prandtl had derived the same results based on similar reasoning already in January 1945.

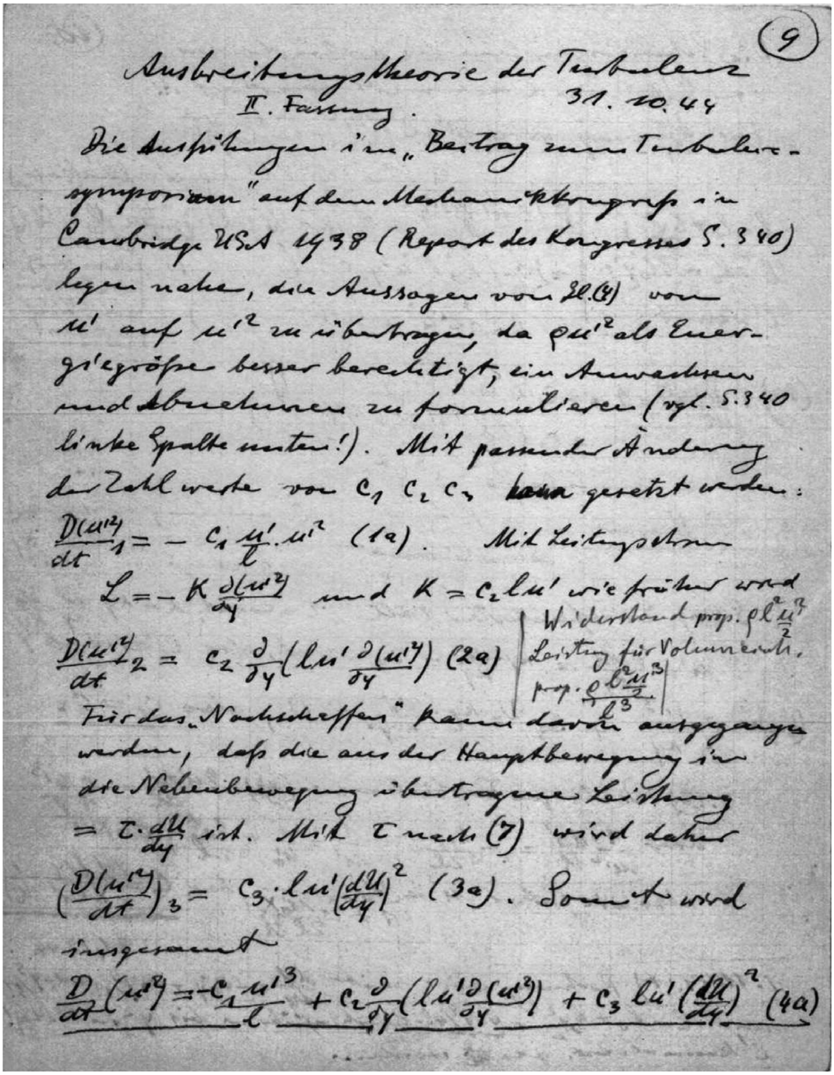
His drafting of the paper fell right into the end of WWII. By July 1945 the Institute was under British administration and had[74] "many British and American visitors". Prandtl was still allowed[75] "to work on some problems that were not finished during the war and from which also reports were expected. Starting any new work was forbidden." By 11 October 1945 the chances for publication were even worse because[76] "all research was shelved completely" and "any continuation of research was forbidden by the Director of Scientific Research in London". So it seems that as of August 1945 Prandtl followed orders and stopped writing the paper and stopped working on turbulence (see Figure 2.8). In addition, in that period the AVA was being disassembled and the parts were sent to England. Then in January 1946, Heisenberg and Weizsäcker returned to Göttingen from being interned in England and brought along their calculations that superseded Prandtl's work. So by January 1946 the window of opportunity for publication had passed. In addition, he was busily writing the FIAT report on turbulence – and that is where he at least mentioned his work.

Let us now discuss briefly Prandtl's last known work on turbulence. His very carefully written notes[77] cover the period from 14 October 1944 until 12 August 1945; they allow us to understand his achievement better. These notes comprise 65 numbered pages, 5 pages on his talk in a Theory Colloquium on 4 January 1945 where he presented his energy equation of turbulence, 7 pages on an attempt at understanding the distribution function of velocity from probability arguments, and 19 pages of sketches and calculations. Figure 2.8 summarizes the days he entered careful handwritten notes in his workbook. From these entries we see that he devoted a large part of his time to the study of turbulence. It seems remarkable how much time he was able to dedicate to this topic, considering that he also directed the research at his Institute and that he was engaged as an adviser to the Air Ministry concerning the direction of aeronautical research for the war. How much effort he dedicated to the latter activity is open to further historical inquiry.

In order to discern the subsequent stages of Prandtl's approach we proceed chronologically:

On 31 October 1944 we find the first complete formulation of the 'one equation model' for the evolution of turbulent kinetic energy per unit volume in terms of the square of the fluctuating velocity (see Figure 2.9). The same

[74] Prandtl to Taylor, 18 July 1945. MPGA, Abt. III, Rep. 61, Nr. 1654.
[75] Prandtl to Taylor, 18 July 1945. MPGA, Abt. III, Rep. 61, Nr. 1654. See also Prandtl to the President of the Royal Society, London, 11 October 1945. MPGA, Abt. III, Rep. 61, Nr. 1402.
[76] Prandtl to Taylor, 18 October 1945. MPGA, Abt. III, Rep. 61, Nr. 1654.
[77] GOAR 3727.



Ausbreitungstheorie der Turbulenz (9)

II. Fassung. 31.10.44

Die Ausführungen im „Beitrag zum Turbulenzsymposium" auf dem Mechanikerkongreß in Cambridge USA 1938 (Report des Kongresses S. 340) legen nahe, die Aussagen von Gl. (8) von u' auf u'^2 zu übertragen, da $\rho u'^2$ als Energiegröße besser berechtigt, ein Anwachsen und Abnehmen zu formulieren (vgl. S. 340 linke Spalte unten!). Mit passender Änderung der Zahlwerte von c_1 c_2 c_3 kann gesetzt werden:

$$\frac{D(u'^2)}{dt}_1 = -c_1 \frac{u'}{l} \cdot u'^2 \quad (1a)$$

Mit Leistungsstrom

$$L = -K \frac{\partial (u'^2)}{\partial y} \text{ und } K = c_2 l u'$$

wie früher wird

$$\frac{D(u'^2)}{dt}_2 = c_2 \frac{\partial}{\partial y}\left(l u' \frac{\partial (u'^2)}{\partial y}\right) \quad (2a)$$

Widerstand prop. $\rho l^2 \frac{u'^2}{2}$; Leistung für Volumeneinh. prop. $\frac{\rho l^2 u'^3}{l^3}$

Für das „Nachschaffen" kann davon ausgegangen werden, daß die aus der Hauptbewegung in die Nebenbewegung übertragene Leistung $= \tau \cdot \frac{dU}{dy}$ ist. Mit τ nach (7) wird daher

$$\left(\frac{D(u'^2)}{dt}\right)_3 = c_3 \cdot l u' \left(\frac{dU}{dy}\right)^2 \quad (3a)$$

Somit wird insgesamt

$$\frac{D}{dt}(u'^2) = -c_1 \frac{u'^3}{l} + c_2 \frac{\partial}{\partial y}\left(l u' \frac{\partial (u'^2)}{\partial y}\right) + c_3 l u' \left(\frac{dU}{dy}\right)^2 \quad (4a)$$

Figure 2.9 Derivation of energy-balance equation without dissipation.

formula now written in terms of kinetic energy per unit volume was presented by him in his paper at the meeting of the Göttingen Academy of Science on 26 January 1945 (Prandtl and Wieghardt, 1945). There Wieghardt also presented his determination of the parameters from measurements in grid turbulence and channel flow and found good agreement with the theory. His differential equation marked as Eq. 4a in Figure 2.9 determined the change of energy per unit volume from three terms: the first term on the right-hand side gives the turbulent energy flux for a "bale of turbulence" (in German: Turbulenzballen) of size l (which he equated with a mixing length), the second term represents the diffusion of turbulence in the direction of the gradient of turbulent energy and the third term represents the source of turbulent energy from the mean shear. Here $cu'l$ is the eddy viscosity. Please note that this equation is what is now called a k-model. This equation was independently derived later by Howard Emmons in 1954 and by Peter Bradshaw in 1967 (Spalding, 1991). Prandtl and Wieghardt also pointed out the deficiencies of the model, namely the role of viscosity at the wall and for the inner structure of a bale of turbulence. For the latter, Prandtl argued that, as long as the Reynolds number of each bale of turbulence is large, a three-dimensional version of his equation should also be applicable to the inner dynamics of the bale of turbulence. He then introduced the idea of bales of turbulence within bales of turbulence, which we now know as the turbulent cascade. He called them steps (in German: Stufen) in the sequence that goes from large to small. He pointed out that by going from step to step the Reynolds number will decrease with increasing number of the step (decreasing size of the turbulent eddy) until viscosity is dominant and all energy is transformed to heat. Finally he conjectured that a general understanding of the process can be obtained.

And indeed he discovered it within a short time. On 29 January 1945, only three days after the presentation at the Academy, he entered in his notes the derivation of the Kolmogorov length scale that at this time he called in analogy to Taylor's smallest length scale λ (Figure 2.10). In Eq. (1) the decay rate of the kinetic energy per unit mass is equated with the dissipation at the smallest scales. Eq. (2) connects the final step of the cascade process with the Kolmogorov velocity. By putting (1) and (2) together Prandtl arrived at the Kolmogorov length scale given by Eq. (3). This seemed to him so remarkable that he commented on the side of the page "Checked it multiple times! But only equilibrium of turbulence."

At this stage he was almost done, but as we can tell from his typewritten manuscript (Prandtl, 1945) and from his notes he was not satisfied. He had to put this on more formal grounds. So only two weeks later, as shown in Figure 2.11 he used a cascade model for each step of turbulence, with β as the ratio in length scale from step to step. This allowed him to derive the Kolmogorov length more rigorously from a geometrical series.

This became the content of his draft paper from 1945 that we shall discuss in detail in a separate publication. It is clear that the spectral data from Motzfeld (1938) were instrumental for his progress (those which Prandtl had presented at the Cambridge Congress in 1938 and which are reproduced above

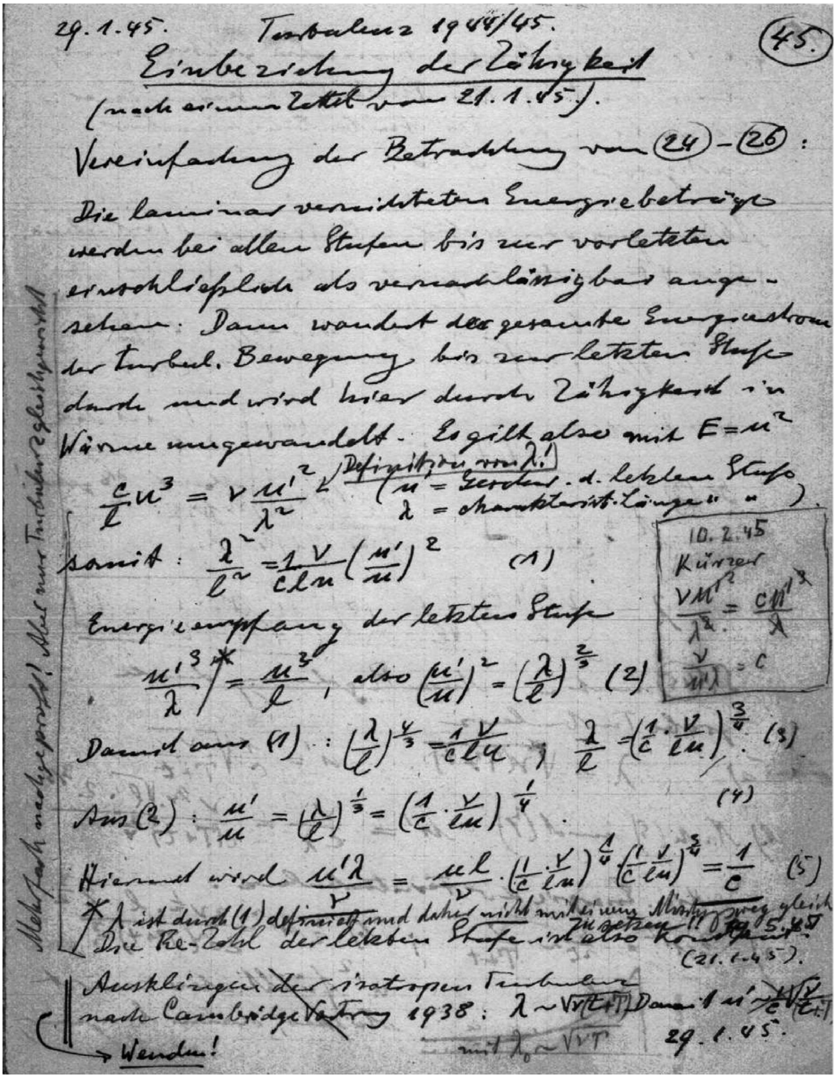


Figure 2.10 First known derivation of the Kolmogorov length scale (here called λ).

in Figure 2.6). Here we close our review with a quote from the introduction to his unpublished paper "The role of viscosity in the mechanism of developed turbulence" (Prandtl, 1945) which beautifully reflects his thinking and needs no further analysis. This is only a short excerpt from the introduction to the paper. A full translation is in preparation and will be published elsewhere. Also,

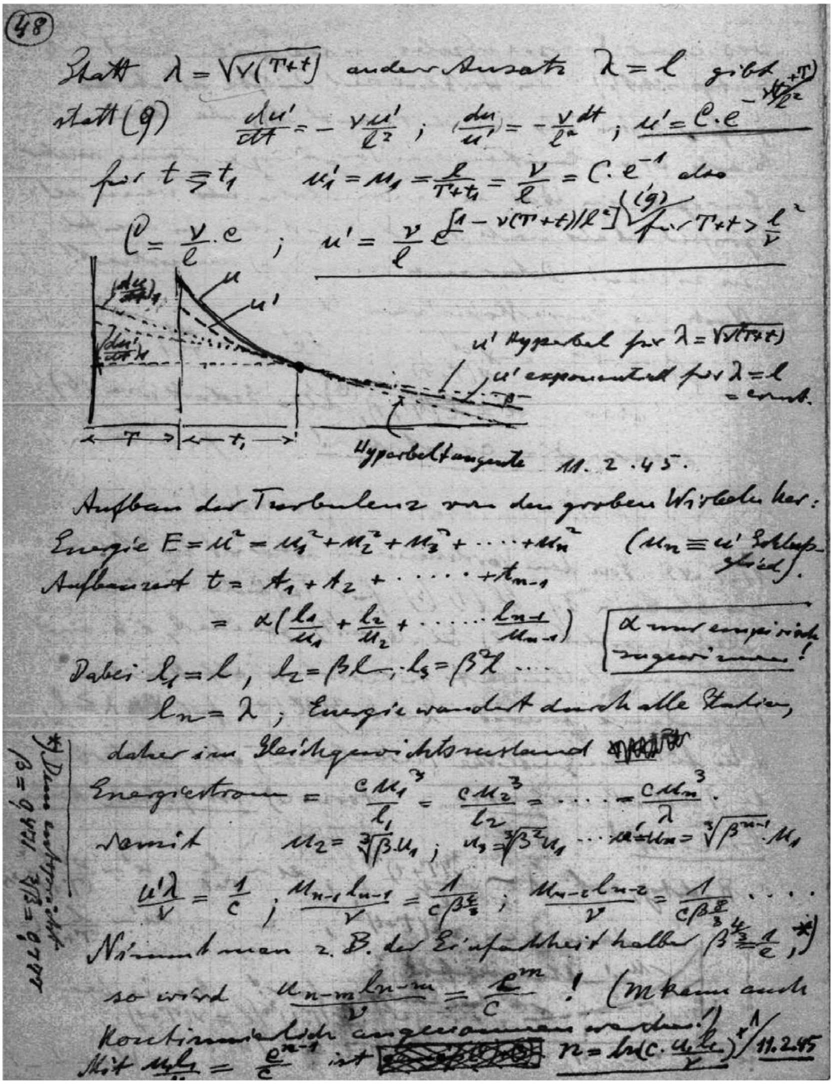


Figure 2.11 Prandtl's cascade model for the fluctuating velocities at different steps in the cascade.

our translation is very close to the original German text and therefore some sentences are a bit long.

The following analyses, which consider in detail the inner processes of a turbulent flow, will prove that the solution for λ by Taylor that he obtained from energy considerations does not yet give the smallest element of turbulence. The mechanism of turbulence generation is not resolved in all details. So much is however known [here Prandtl referred in a footnote to work by Tollmien published in *Göttinger Nachr*. Heft 1 (1935) p. 79] that flows with an inflection point in the velocity profile may become unstable at sufficiently large Reynolds numbers. Therefore one has to expect that at sufficiently high Reynolds number $\frac{u'l}{\nu}$ the motion of an individual bale of turbulence is by itself turbulent, and that for this secondary turbulence the same is true, and so on. Indeed one observes already at very modest Reynolds numbers a frequency spectrum that extends over many decades. That it is mostly the smallest eddies that are responsible for the conversion of the energy of main motion into heat can easily be understood, as for them, the deformation velocities $\frac{\partial u}{\partial y}$, etc. are the largest.

The earlier discussion is the simplest explanation of the fact that in turbulent motion always the smaller eddies are present next to the larger ones. G.I. Taylor, 1935, used a different explanation. He pointed out that according to general statistical relations the probability of two particles separating in time is larger than for them to come closer, and he applied this relationship to two particles on a vortex line. From the well-known Helmholtz theorem it would follow that – as long as the viscosity does not act in an opposing sense – the increase of the angular velocity of the vortex line is more probable than its decay. He shows this tendency with an example, whose series expansion clearly shows the evolution towards smaller eddies. However, it could not be continued, so the processes could only be followed for short time intervals. One can counter Taylor's deductions insofar that through the increase of the angular velocities, pressure fields develop, which oppose a further increase of the vortex lines. It thus cannot be expected that the extension would reach the expected strength. It seems, however, that the action in the sense of Taylor is surely present, if, though, with weaker magnitude than expected from a purely kinematic study.

For the development of smaller eddy diameters in the turbulence, one can also note that wall turbulence starts with thin boundary layers and that free turbulence has equally thin separating sheets. Therefore, in the beginning, only the smallest vortices are present and the larger ones appear one after another. Opposing this, however, is the result that in the fully developed channel flow, the frequency spectrum *de facto* does not depend on the distance from the wall (Motzfeld, 1938). One would not expect this if all of the fine turbulence originated at the wall. This strongly supports the validity of the conjecture for stationary turbulence presented here. Further support is given by investigations conducted later, which concerned isotropic, temporally decaying, turbulence and which have been quite satisfactorily justified by experiments. The two descriptions of the re-creation of the smaller eddies by turbulence of second and higher order, and the one that relates to the Helmholtz theorem, are, by the way, intricately related: they are both, so to say, descriptions that elucidate one and the same process only from different perspectives.

In the following, initially temporally stationary turbulence may be assumed, as is found, for example, in a stationary channel or pipe flow. Of the dissipated power D in a unit volume per unit time, a very small fraction $\mu(\frac{\partial U}{\partial y})^2$ will be dissipated immediately into heat (U is the velocity of the mean flow); the rest, which one may call D_1, increases the kinetic energy of the turbulent submotion [Nebenbewegung] and generates, according to Taylor, secondary turbulence . . .

[Here we leave out some equations.]

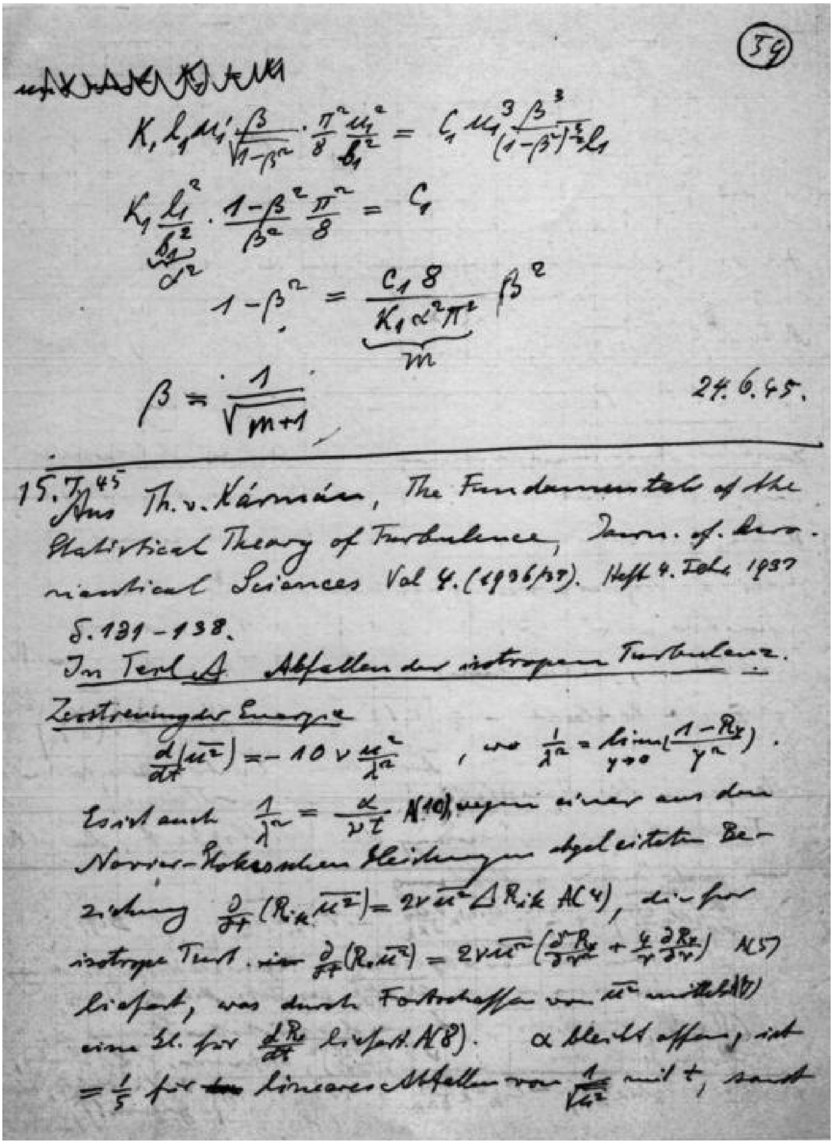
We now establish corresponding equations for turbulence of the second step (third step etc.). Instead of the velocity U, here a suitably smoothed velocity of turbulence of first step, second step etc. must be used. The instantaneous values of the velocity u, which is used as a representative for the triple (u, v, w) for simplicity, will thus be separated into a sum of partial velocities of which u_1 is the smoothest main part of u and represents the 'first step'; correspondingly, the smaller, but finer-structured part, u_2, the second step etc.; the nth order shall be the last one in the series that will no longer become turbulent [here Prandtl added in a footnote: "The separation into steps thereby creates difficulties, namely that the elements of the first step do not all have the same size and that in the following the differences may increase even further. As the purpose of the analysis is only a rough estimate one may conjecture that the elements in each step have the same well-defined size. A more detailed analysis by considering the statistical ensemble of turbulence elements is an aim for the future."]

Motivated by the way the u_i are introduced, it seems natural to assume that their effective values u_i^2 build a geometric series, at least with the exclusion of the final members of the series, for which viscosity is already noticeable. As a first approximation one may assume also that the final members of the series, other than the very last one, are members of the geometric series.

By this reasoning Prandtl ended the geometric series by closing it with a single last step at which all energy dissipation occurs. His final derivation of the Kolmogorov length scale is then quite similar to what he calculated on 29 January 1945 (see Figure 2.10).

This did not conclude Ludwig Prandtl's quest for an understanding of turbulence. In mid July 1945 he had realized that his 'one equation model' was missing a second equation that allowed him to determine the mixing length. Therefore he resorted to the vorticity equation that Kármán had investigated. As shown in Figures 2.12–2.15 he calculated with help of his vorticity equation B9 (see Figure 2.13) for the case of plane shear flow under the assumption of 'homologue' turbulence (for which the correlation coefficients of the velocity components are independent of space) the mixing length and $\frac{dU}{dy}$.

So from October 1944 to August 1945, Prandtl had returned to his lifelong quest to understand turbulence. On 17 September 1945 the Georg August University was re-opened as the first in post-war Germany and Prandtl taught again. By January 1946, Otto Hahn, Werner Heisenberg, Max von Laue and Carl Friedrich von Weizsäcker returned from England to Prandtl's Institute that was reopened on 1 August 1946. Max Planck became interim President of the Kaiser-Wilhelm-Society, which now had its headquarters in the buildings of Prandtl's Institute. On 11 September 1946 the Max Planck Society was founded in Bad Driburg as the successor of the Kaiser-Wilhelm-Society. On


$$K_1 l_1 u_1 \frac{\beta}{\sqrt{1-\beta^2}}\cdot\frac{\pi^2}{8}\frac{u_1^2}{b_1^2} = C_1 u_1^3 \frac{\beta^3}{(1-\beta^2)^{\frac{3}{2}} l_1}$$

$$K_1 \frac{l_1^2}{b_1^2}\cdot\frac{1-\beta^2}{\beta^2}\frac{\pi^2}{8} = C_1$$

$$1-\beta^2 = \underbrace{\frac{C_1 8}{K_1\alpha^2\pi^2}}_{m}\beta^2$$

$$\beta = \frac{1}{\sqrt{m+1}}$$

24.6.45.

15.7.45 Th. v. Kármán, The Fundamentals of the Statistical Theory of Turbulence, Journ. of Aeronautical Sciences Vol 4. (1936/37). Heft 4. Febr. 1937 S. 131–138.

Im Text A Abfallen der isotropen Turbulenz. Zerstreuung der Energie

$$\frac{d}{dt}(\overline{u^2}) = -10\nu\frac{u^2}{\lambda^2}, \quad \text{wo } \frac{1}{\lambda^2} = \lim_{y\to 0}\left(\frac{1-R_y}{y^2}\right).$$

Figure 2.12 Calculation of the mixing length from the vorticity equation; 1 of 4.

26 February 1948 the Max Planck Society convened its constitutional meeting in the cafeteria of Prandtl's Institute.

Prandtl himself retired from the University and Institute's Directorships in the fall of 1946 and continued working on problems in meteorology until his death on 15 August 1953.

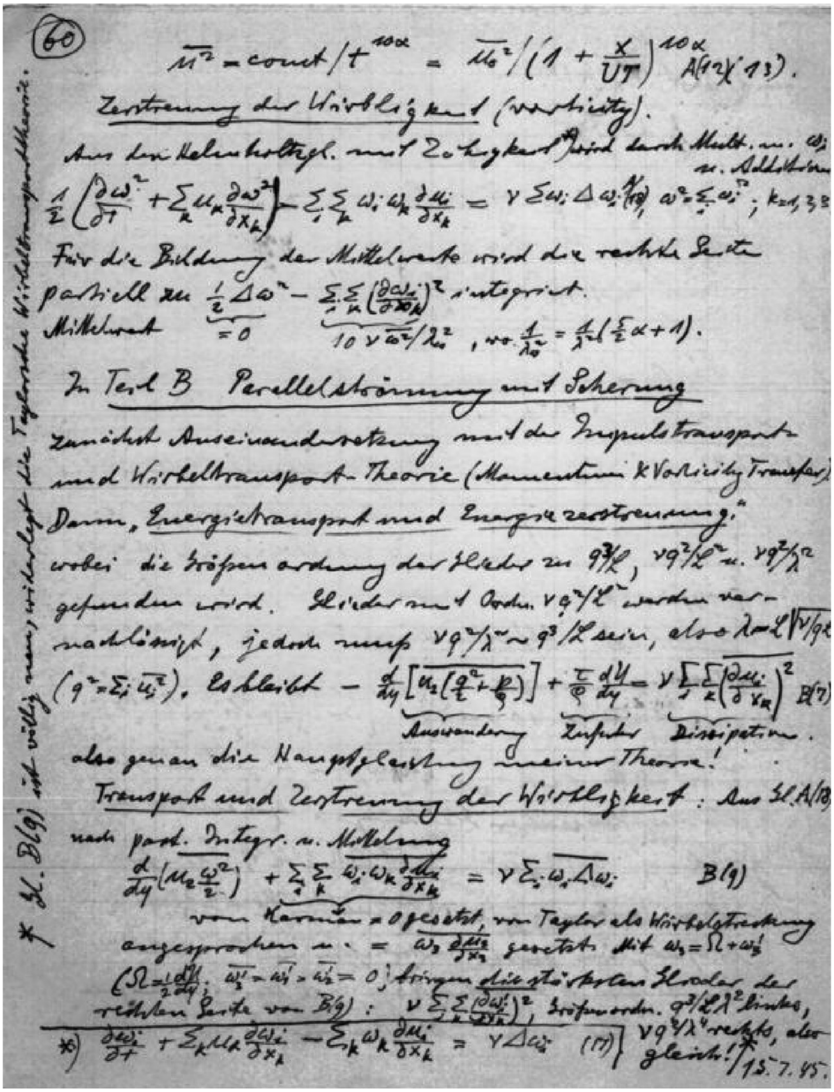
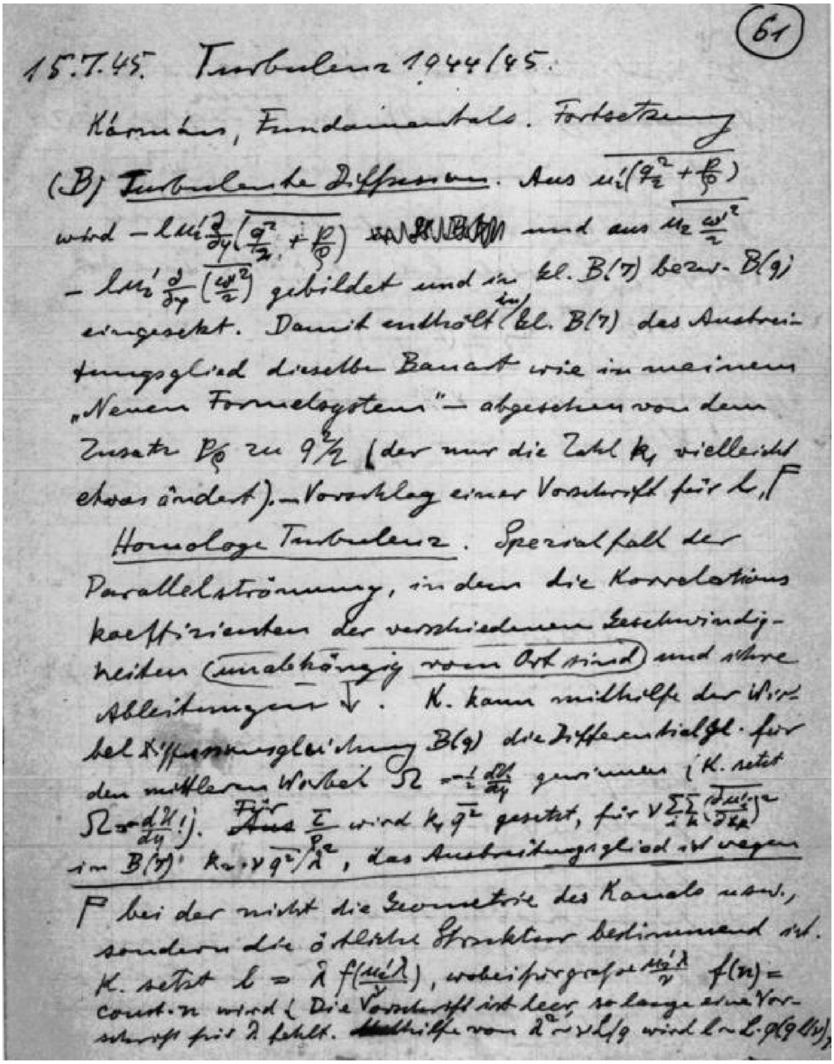


Figure 2.13 Calculation of the mixing length from the vorticity equation; 2 of 4.

2.11 Conclusion

Prandtl's achievements in fluid mechanics generally, and in turbulence in particular, are often characterized by the label 'theory'. However, it is important to note that he did not perceive himself as a theoretician. When the German Physical Society of the British Zone awarded him honorary membership two years


61

15.7.45. Turbulenz 1944/45.

Kármán, Fundamentals. Fortsetzung

(B) Turbulente Diffusion. Aus $\overline{u_i'(q^2/2 + p/\rho)}$ wird $-l u_i' \frac{\partial}{\partial y}\overline{(q^2/2 + p/\rho)}$ und aus $\overline{u_2 \omega'^2/2}$ $-l u_2' \frac{\partial}{\partial y}\overline{(\omega'^2/2)}$ gebildet und in Gl. B(7) bezw. B(9) eingesetzt. Damit enthält in Gl. B(7) das Ausbreitungsglied dieselbe Bauart wie in meinem „Neuen Formelsystem" – abgesehen von dem Zusatz p/ρ zu $q^2/2$ (der nur die Zahl k_1 vielleicht etwas ändert). – Vorschlag einer Vorschrift für l, [illegible]

Homologe Turbulenz. Spezialfall der Parallelströmung, in dem die Korrelationskoeffizienten der verschiedenen Geschwindigkeiten (unabhängig vom Ort sind) und ihre Ableitungen ↓. K. kann mithilfe der Wirbel Diffusionsgleichung B(9) die Differentialgl. für den mittleren Wirbel $\Omega = -\frac{1}{2}\frac{dU}{dy}$ gewinnen (K. setzt $\Omega = \frac{dU}{dy}$!). Für $\frac{\tau}{\rho}$ wird $k_1 \overline{q^2}$ gesetzt, für $\nu \sum\sum \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2}$ in B(7) $k_2 \nu \overline{q^2}/\lambda^2$, das Ausbreitungsglied ist wegen

[illegible] bei der nicht die Geometrie des Kanals usw., sondern die örtliche Struktur bestimmend ist. K. setzt $l = \lambda f\left(\frac{u_i'\lambda}{\nu}\right)$, wobei für große $\frac{u_i'\lambda}{\nu}$ $f(n) =$ const. $\cdot n$ wird (Die Vorschrift ist leer, so lange eine Vorschrift für λ fehlt. Mithilfe von $\lambda^2 \sim \nu L/q$ wird $l \sim L \cdot \varphi(qL/\nu)$,

Figure 2.14 Calculation of the mixing length from the vorticity equation; 3 of 4.

after the war, he used this occasion to clarify his research style in a lecture entitled "My approach towards hydrodynamical theories". With regard to boundary layer theory, for example, he argued that he was guided by a 'heuristic principle' of this kind: "If the whole problem appears mathematically hopeless, see what happens if an essential parameter of the problem approaches zero" (Prandtl, 1948b, p. 1606). His notes amply illustrate how he used one

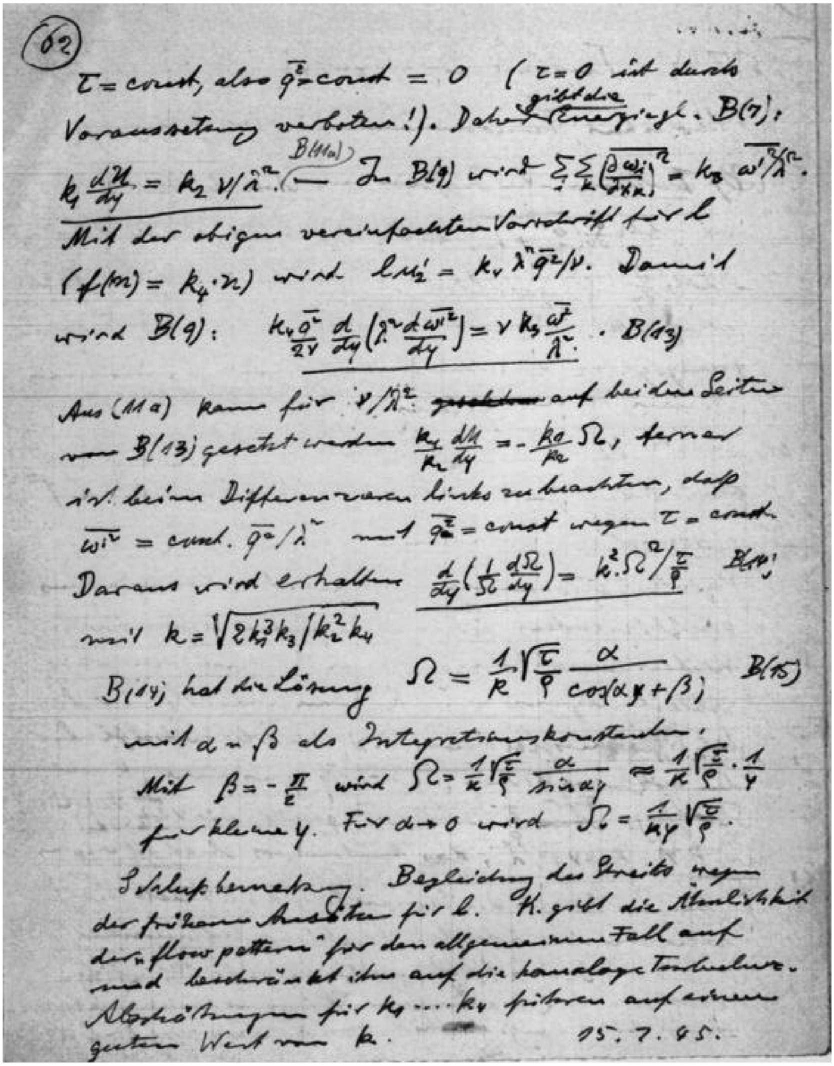


Figure 2.15 Calculation of the mixing length from the vorticity equation; 4 of 4.

or another assumption, often combined with clever dimensional arguments, in order to single out those features of a problem which he regarded as crucial. He always attempted to gain "a thorough visual impression" about the problems with which he was concerned. "The equations come later when I think that I have grasped the matter" (Prandtl, 1948b, p. 1604).

By the same token, Prandtl's approach to theory relied heavily on practice. For that matter, practice could be an observation of flow phenomena in a water channel, an *experimentum crucis* like the trip-wire test, or a challenge posed by practical applications such as skin friction. Prandtl's FIAT review on turbulence, in particular, illustrates how his theoretical research was motivated and guided by practice. As we have seen, Prandtl named explicitly, among others, Schultz-Grunow (1940), Wieghardt (1944), Reichardt (1944) and Schuh (1945) as important roots for the theoretical insight expressed in Prandtl and Wieghardt (1945). His closest collaborator for the fundamental studies on fully developed turbulence, Wieghardt, was by that time developing technical expertise for studying the skin friction of rubber with regard to a possible use for the hull of submarines (Prandtl, 1948a, p. 58). These and other war-related studies were based on experimental turbulence measurements in the same 'roughness tunnel' that provided the data for the more fundamental inquiries.

Prandtl's style as well as the closeness of theory and practice is also reflected in the third edition of his famous *Essentials of Fluid Mechanics* (Prandtl, 1948c). In a paragraph about the onset of turbulence, for example, Prandtl reported about the recent confirmation of the Tollmien–Schlichting theory by the experiments in Dryden's laboratory at the National Bureau of Standards in Washington. Turbulent jets and turbulent shear flow along walls were discussed in terms of the mixing length approach (Prandtl, 1948c, pp. 115–123). Isotropic turbulence was summarized rather cursorily, with a reference to his FIAT review and the recent work by Weizsäcker and Heisenberg (Prandtl, 1948c, p. 127). In general, he preferred textual and pictorial presentations supported by experiments over sophisticated mathematical derivations.

For a deeper understanding of Prandtl's and his Göttingen school's contributions to turbulence it would be necessary to account for the broader research conducted at the KWI and the AVA, which covered a host of fundamental and applied topics, from solid elasticity to gas dynamics and meteorology. Research on turbulence was never pursued as an isolated topic. But in view of its ultimate importance for engineering, turbulence always remained an important and challenging problem. Among the variety of research problems dealt with at Göttingen in the era of Prandtl, turbulence may be regarded as the one with the longest tradition – from Klein's seminar in 1907 to the climax of Prandtl's unpublished manuscripts in 1945.

A number of questions have been left unanswered. The timing of Prandtl's breakthrough during the last months of the Second World War, in particular, suggests further inquiries: to what extent was fundamental research on turbulence interrupted during the war by Prandtl's involvement as a Scientific Adviser to the Ministry of Aviation (Reichsluftfahrtsministerium) with regards to aeronautical war research?[78] Or was the renewed interest in the basic riddles of turbulence sparked by the wartime applications? Or, on the contrary, did Prandtl at the end of the war find the time to work on what he was really interested in?

Both Prandtl's advisory role as well as his local responsibilities for fluid dynamics research at Göttingen came to a sudden stop when the American and British troops occupied his Institute and prevented further research – a prohibition which Prandtl perceived as unwarranted. Not only did he write[79] to Taylor for help, but also he requested[80] help from the President of the Royal Society, of which he had been a Foreign Member since 1928. "The continuation of the research activity that had to be shelved during the War should not be hindered any more!" demanded Prandtl in this letter. His request remained unanswered.

This correspondence provokes further questions regarding Prandtl's political attitude. Biographical knowledge of Prandtl has been provided by his family (Vogel-Prandtl, 1993), by admiring disciples (Flügge-Lotz and Flügge, 1973; Oswatitsch and Wieghardt, 1987), and by reviews on German wartime aeronautical research (Trischler, 1994); a more complete view based on the rich sources preserved in the archives in Göttingen, Berlin and elsewhere seems expedient.[81] Recent historical studies on the war research at various Kaiser-Wilhelm-Institutes (see, for example, Maier, 2002; Schmaltz, 2005; Sachse and Walker, 2005; Maier, 2007; Heim et al., 2009; Gruss and Ruerup, 2011) call for further inquiries into Prandtl's motivations for research into turbulence. An important question of course is: What can we learn from the position of great men like Prandtl and others in the political web of Nazi Germany? What consequences arise for the responsibilities of scientists or engineers? Another lacuna which needs to be addressed in greater detail concerns the relationship of Prandtl with his colleagues abroad and in Germany, in particular with von Kármán, Taylor, Sommerfeld and Heisenberg. Last, but not least, one may ask about the fate of turbulence research at Göttingen under Prandtl's successors after the war. We leave these and many other questions for future studies.

Acknowledgement It is our pleasure to express our gratitude to a large number of people without whom this work would not have been possible. We are very thankful to the science historian Florian Schmaltz for sharing his insights and for giving us copies of documents from his collections. We express

[78] From 6 July 1942 Prandtl became the Chair of the Scientific Research Council of the Ministry of Aviation led by Göring and was pushing for fundamental research in war-related matters (Maier, 2007).

[79] Prandtl to Taylor, 18 October 1945. MPGA, Abt. III, Rep. 61, Nr. 1654.

[80] Prandtl to Royal Society, 18 October 1945. MPGA, Abt III, Rep. 61, Nr. 1402.

[81] An almost complete set of his correspondence is preserved.

our gratitude to the Max Planck Society Archives, especially Lorenz Beck, Bernd Hofmann, Susanne Übele and Simone Pelzer; and to the Archives of the German Aerospace Center (DLR), especially Jessika Wichner and Andrea Missling. We are also very grateful to K. Sreenivasan, N. Peters and G. Falkovich for sharing their insights on this topic and to the editors of this book for motivating us to write this review and helpful suggestions for improvements of the text. EB also thanks Haitao Xu and Zellman Warhaft for valuable comments, and Helmut Eckelmann for sharing his memories about the Institute. We are most grateful for the understanding and support from our families that were missing their husband and father for long evenings and weekends. This work was generously supported by the Max Planck Society and the Research Institute of the German Museum in Munich. Part of this work was written at the Kavli Institute for Theoretical Physics and was supported in part by the National Science Foundation under Grant No. NSF PHY05-51164.

Abbreviations

DMA: Deutsches Museum, Archiv, München.

GOAR: Göttinger Archiv des DLR, Göttingen.

LPGA: Ludwig Prandtls Gesammelte Abhandlungen, herausgegeben von Walter Tollmien, Hermann Schlichting und Henry Görtler. 3 Bände, Berlin u. a. 1961.

MPGA: Max-Planck-Gesellschaft, Archiv, Berlin.

RANH: Rijksarchief in Noord-Holland, Haarlem.

SUB: Staats- und Universitätsbibliothek, Göttingen.

TKC: Theodore von Kármán Collection, Pasadena.

References

Anderson, John D. 2005. Ludwig Prandtl's boundary layer. *Physics Today*, **58** 42–48.

Batchelor, G.K. 1946. Double velocity correlation function in turbulent motion. *Nature*, **158** 883–884.

Battimelli, Giovanni. 1984. The mathematician and the engineer: statistical theories of turbulence in the 20's. *Rivista di storia della scienza*, **1** 73–94.

Blasius, Heinrich. 1908. Grenzschichten in Flüssigkeiten bei kleiner Reibung. *Zeitschrift für Mathematik und Physik*, **56** 1–37.

Blasius, Heinrich. 1913. Das Ähnlichkeitsgesetz bei Reibungsvorgängen in Flüssigkeiten. *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, 131.

Boussinesq, M.J. 1897. *Theorie de l'Ecoulement Tourbillonnant et Tumultueux des Liquides dans les Lits Rectilignes á Grandes Sections*. Gauthiers-Villars et fils, Paris.

Burgers, J.M. 1925. The motion of a fluid in the boundary layer along a plane smooth surface. *Proceedings of the First International Congress for Applied Mechanics, Delft, 1924*, C.B. Biezeno and J.M. Burgers (eds), 113–128.

Collar, A.R. 1978. Arthur Fage. 4 March 1890–7 November 1977. *Biographical Memoirs of Fellows of the Royal Society*, 33–53.

Comte-Bellot, G. and Corrsin, S. 1966. The use of a contraction to improve the isotropy of grid-generated turbulence. *J. Fluid Mech.*, **25** 667–682.

Darrigol, Olivier. 2005. *Worlds of Flow*. Oxford University Press, Oxford.

Dryden, Hugh L. and Kuethe, A.M. 1929. The measurement of fluctuations of air speed by the hot-wire anemometer. *NACA Report*, **320** 357–382.

Dryden, Hugh L., Schubauer, G.B., Mock, W.C. and Skramstadt, H.K. 1937. Measurements of intensity and scale of wind-tunnel turbulence and their relation to the critical Reynolds number of spheres. *NACA Report*, **581** 109–140.

Dryden, Hugh L. 1938. Turbulence investigations at the National Bureau of Standards. *Proceedings of the Fifth International Congress on Applied Mechanics, Cambridge Mass.*, J.P. Den Hartog and H. Peters (eds), Wiley, New York, 362–368.

Dryden, Hugh L. 1955. Fifty years of boundary-layer theory and experiment. *Science*, **121** 375–380.

Eckert, Michael. 2006. *The Dawn of Fluid Dynamics*. Wiley-VCH, Weinheim.

Eckert, Michael. 2008. Theory from wind tunnels: empirical roots of twentieth century fluid dynamics. *Centaurus*, **50** 233–253.

Eckert, Michael. 2010. The troublesome birth of hydrodynamic stability theory: Sommerfeld and the turbulence problem. *European Physical Journal, History*, **35**(1) 29–51.

Eiffel, Gustave. 1912. Sur la résistance des sphères dans l'air en mouvement. *Comptes Rendues*, **155** 1597–1599.

Eisner, F. 1932a. Reibungswiderstand. *Werft, Reederei, Hafen*, **13** 207–209.

Eisner, F. 1932b. Reibungswiderstand. In *Hydromechanische Probleme des Schiffsantriebs. Hamburg*, G. Kempf and E. Foerster (eds), 1–49.

Fage, A. and Townend, H.C.H. 1932. An examination of turbulent flow with an ultramicroscope. *Proc. Roy. Soc. Lond. A*, **135** 656–677.

Flüigge-Lotz, Irmgard and Flügge, Wilhelm. 1973. Ludwig Prandtl in the nineteen-thirties: reminiscences. *Ann. Rev. Fluid Mech.* **5** 1–8.

Föppl, Otto. 1912. Ergebnisse der aerodynamischen Versuchsanstalt von Eiffel, verglichen mit den Göttinger Resultaten. *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, **3** 118–121.

Fritsch, Walter. 1928. Der Einfluss der Wandrauhigkeit auf die turbulente Geschwindigkeitsverteilung in Rinnen. *Abhandlungen aus dem Aerodynamischen Institut der Technischen Hochschule Aachen*, **8**.

Gilbert, Martin. 2006. *Kristallnacht: Prelude to Destruction*. Harper Collins Publishers.

Görtler, H. 1942. Berechnungen von Aufgaben der freien Turbulenz auf Grund eines neuen Näherungsansatzes. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **22** 44–254.

Goldstein, Sydney. 1969. Fluid mechanics in the first half of this century. *Ann. Rev. Fluid Mech.*, **1** 1–28.

Grossmann, Siegfried, Eckhardt, Bruno and Lohse, Detlef. 2004. Hundert Jahre Grenzschichtphysik. *Physikjournal*, **3** (10), 31–37.

Gruschwitz, E. 1931. Die trubulente Reibungsschicht bei Druckabfall und Druckanstieg. *Ingenieur-Archiv*, **2** 321–346.

Gruss, Peter and Rürup, Reinhard (eds). 2011. *Denkorte. Max-Planck-Gesellschaft und Kaiser-Wilhelm-Gesellschaft, Brüche und Kontinuitäten 1911–2011*. Sandstein Verlagr., Dresden.

Hager, W.H. 2003. Blasius: A life in research and education. *Experiments in Fluids*, **34** 566–571.

Hahn, Hans, Herglotz, Gustav and Schwarzschild, Karl, 1904. Über das Strömen des Wassers in Röhren und Kanälen. *Zeitschrift für Mathematik und Physik*, **51** 411–426.

Hamel, H. 1943. Streifenmethode und Ähnlichkeitsbetrachtungen zur turbulenten Bewegung. *Abh. Preuss. Akad. Wiss. Physik.–Math.*, **8**.

Heim, Susanne, Sachse, Carola and Walker, Mark (eds). 2009. *The Kaiser Wilhelm Society under National Socialism*. Cambridge University Press, Cambridge.

Heisenberg, W. 1948. Zur statistischen Theorie der Turbulenz. *Zeitschrift für Physik*, **124** 628–657.

Heisenberg, W. 1958E. On the statistical theory of turbulence. NACA-TM-1431, 1958.

Hensel, Susann. 1989. *Mathematik und Technik im 19. Jahrhundert in Deutschland. Soziale Auseinandersetzungen und philosophische Problematik*, chapter Die Auseinandersetzungen um die mathematische Ausbildung der Ingenieure an den Technischen Hochschulen in Deutschland Ende des 19. Jahrhunderts, 1–111. Vandenhoeck und Ruprecht, Göttingen.

Hopf, Ludwig. 1910. *Hydrodynamische Untersuchungen: Turbulenz bei einem Flusse. Über Schiffswellen. Inaugural-Dissertation*. Barth, Leipzig.

Jakob, M. and Erk, S. 1924. Der Druckabfall in glatten Rohren und die Durchflussziffer von Normaldüsen. *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, **267**.

Kármán, von, Theodore. 1921. Über laminare und turbulente Reibung. *ZAMM*, **1** 233–252. CWTK 2, 70–97.

Kármán, von, Theodore. 1924. Über die Oberflächenreibung von Flüssigkeiten. In *Vorträge aus dem Gebiete der Hydro- und Aerodynamik (Innsbruck 1922)*. Theodore von Kármán und T. Levi-Civita (eds). Berlin: Springer, 146–167. CWTK 2, 133–152.

Kármán, von, Theodore. 1930a. Mechanische Ähnlichkeit und Turbulenz. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch–Physikalische Klasse*, 58–76. CWTK 2, 322–336.

Kármán, von, Theodore. 1930b. Mechanische Ähnlichkeit und Turbulenz. In *Proceedings of the Third International Congress of Applied Mechanics, Stockholm*. CWTK 337–346.

Kármán, von, Theodore. 1932. Theorie des Reibungswiderstandes. In *Hydromechanische Probleme des Schiffsantriebs. Hamburg*, G. Kempf, E. Foerster (eds), 50–73. CWTK 2, 394–414.

Kármán, von, Theodore. 1934. Turbulence and skin friction. *Journal of the Aeronautical Sciences*, **1** 1–20. CWTK 3, 20–48.

Kármán, von, Theodore. 1957. *Aerodynamics: Selected Topics in the Light of their Historical Development*. Cornell University Press, Ithaca, New York.

Kármán, von, Theodore (with Lee Edson). 1967. *The Wind and Beyond*. Little, Brown and Company, Boston.

Kármán, von, Theodore and Leslie Howarth. 1938. On the statistical theory of isotropic turbulence. *Proc. Roy. Soc. Lond. A*, **164** 192–215.

Kempf, Günther. 1929. Neue Ergebnisse der Widerstandsforschung. *Werft, Reederei, Hafen*, **10** 234–239.

Kempf, Günther. 1932. Weitere Reibungsergebnisse an ebenen glatten und rauhen Flächen. *Hydromechanische Probleme des Schiffsantriebs. Hamburg*, G. Kempf, E. Foerster (eds), 74–82.

Klein, Felix. 1910. Über die Bildung von Wirbeln in reibungslosen Flüssigkeiten. *Zeitschrift für Mathematik und Physik*, **59** 259–262.

Kuethe, A.M. 1988. The first turbulence measurements: a tribute to Hugh L. Dryden. *Ann. Rev. Fluid Mech.*, **20** 1–3.

Lorentz, Hendrik Antoon. 1897. Over den weerstand dien een vloeistofstroom in eene cilindrische buis ondervindt. *Versl. K. Akad. Wet. Amsterdam*, **6** 28–49.

Lorentz, Hendrik Antoon. 1907. Über die Entstehung turbulenter Flüssigkeitsbewegungen und über den Einfluss dieser Bewegungen bei der Strömung durch Röhren. *Hendrik Antoon Lorentz: Abhandlungen über theoretische Physik*. Teubner, Leipzig, **1** 43–71.

Lumley, John Leask and Panofsky, Hans A. 1964. *The Structure of Atmospheric Turbulence*. Wiley, New York.

Maier, Helmut (ed.). 2002. *Rüstungsforschung im Nationalsozialismus. Organisation, Mobilisierung und Entgrenzung der Technikwissenschaften*. Wallstein, Göttingen, 2002.

Maier, Helmut. 2007. *Forschung als Waffe. Rüstungsforschung in der Kaiser-Wilhelm-Gesellschaft und das Kaiser-Wilhelm-Institut für Metallforschung 1900–1945/48*. Wallstein, Göttingen, 2007.

Manegold, Karl-Heinz. 1970. *Universität, Technische Hochschule und Industrie: Ein Beitrag zur Emanzipation der Technik im 19. Jahrhundert unter besonderer Berücksichtigung der Bestrebungen Felix Kleins*. Duncker und Humblot, Berlin.

Meier, Gerd E.A. 2006. Prandtl's boundary layer concept and the work in Göttingen. *IUTAM Symposium on One Hundred Years of Boundary Layer Research. Proceedings of the IUTAM Symposium held at DLR-Göttingen, Germany, August 12–14, 2004*. G.E.A. Meier and K.R. Sreenivasan (eds). Springer, Dordrecht, 1–18.

Mises, von, Richard. 1921. Über die Aufgaben und Ziele der angewandten Mathematik. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **1** 1–15.

Motzfeld, H. 1938. Vorträge aus dem Gebiet der Aero- und Hydrodynamik. Frequenzanalyse turbulenter Schwankungen. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **18** 362–365.

Munk, Max. 1917. Bericht über Luftwiderstandsmessungen von Streben. Mitteilung 1 der Göttinger Modell-Versuchsanstalt für Aerodynamik. *Technische Berichte. Herausgegeben von der Flugzeugmeisterei der Inspektion der Fliegertruppen*. Heft Nr. 4 (1. Juni 1917), 85–96, Tafel XXXX–LXIII.

Nagib, Hassan M., Chauhan, Kapil A. and Monkewitz, Peter A. 2007. Approach to an asymptotic state for zero pressure gradient turbulent boundary layers. *Phil. Trans. R. Soc. A*, **365** 755–770.

Nikuradse, Johann. 1926. Untersuchungen über die Geschwindigkeitsverteilung in turbulenten Strömungen. *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, **281**.

Nikuradse, Johann. 1930. Über turbulente Wasserströmungen in geraden Rohren bei sehr grossen Reynoldsschen Zahlen. *Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete (Aachen 1929)*. A. Gilles, L. Hopf and Th. v. Kármán (eds). Springer, Berlin, 63–69.

Nikuradse, Johann. 1932. Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren. *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, 356.

Nikuradse, Johann. 1933. Strömungsgesetze in rauhen Rohren. *Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, 3611.

Noether, Fritz. Das Turbulenzproblem. *Zeitschrift für Angewandte Mathematik und Mechanik*, **1** 125–138, 218–219.

O'Malley Jr., Robert E. 2010. Singular perturbation theory: a viscous flow out of Göttingen. *Ann. Rev. Fluid Mech.*, **42** 1–17.

Oswatitsch, K. and Wieghardt, K. 1987. Ludwig Prandtl and his Kaiser-Wilhelm-Institut. *Ann. Rev. Fluid Mech.*, **19** 1–26.

Prandtl, Ludwig. 1905. Über Flüssigkeitsbewegung bei sehr kleiner Reibung. In *Verhandlungen des III. Internationalen Mathematiker-Kongresses, Heidelberg*, 484–491. LPGA 2, 575–584.

Prandtl, Ludwig. 1910. Eine Beziehung zwischen Wärmeaustausch und Strömungswiderstand der Flüssigkeiten. *Physikalische Zeitschrift*, **11** 1072–1078.

Prandtl, Ludwig. 1914. Der Luftwiderstand von Kugeln. *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch–Physikalische Klasse*, 177–190. (LPGA 2, 597–608).

Prandtl, Ludwig. 1921a. Bemerkungen über die Entstehung der Turbulenz. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **1** 431–436.

Prandtl, Ludwig. 1921b. *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*. Oldenbourg, Berlin, München.

Prandtl, Ludwig. 1922. Bemerkungen über die Entstehung der Turbulenz. *Physikalische Zeitschrift*, **23** 19–25.

Prandtl, Ludwig. 1925. Bericht über Untersuchungen zur ausgebildeten Turbulenz. *ZAMM*, **5** 136–139. LPGA 2, 714–718.

Prandtl, Ludwig. 1925E. Aufgaben der Strömungsforschung: Tasks of air flow research. *Die Naturwissenschaften*, **14** (16) 355–358, 1925; NACA–TN–365, 1926.

Prandtl, Ludwig. 1926a. Bericht über neuere Turbulenzforschung. *Hydraulische Probleme*. VDI-Verlag, Berlin, 1–13. LPGA 2, 719–730.

Prandtl, Ludwig. 1926b. Klein und die Angewandten Wissenschaften. *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, 81–87. LPGA 2, 719–730.

Prandtl, Ludwig. 1927a. Über den Reibungswiderstand strömender Luft. *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*. Oldenbourg, Berlin, München, **3** 1–5. LPGA 2, 620-626.

Prandtl, Ludwig. 1927b. Über die ausgebildete Turbulenz. *Verhandlungen des II. Internationalen Kongresses für Technische Mechanik*. Füssli, Zürich, 62–75. LPGA 2, 736–751. NACA-TM-435, version in English.

Prandtl, Ludwig. 1930. Vortrag in Tokyo. *Journal of the Aeronautical Research Institute*, Tokyo, Imperial University, **5** (65) 12–24. LPGA 2, 788–797.

Prandtl, Ludwig. 1931. *Abriss der Strömungslehre*. Vieweg, Braunschweig.

Prandtl, Ludwig. 1932. Zur turbulenten Strömung in Rohren und längs Platten. In *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, **4** 18–29. LPGA 2, 632–648.

Prandtl, Ludwig. 1933. Neuere Ergebnisse der Turbulenzforschung. *Zeitschrift des Vereines Deutscher Ingenieure* **77** 105–114. LPGA 2, 819–845.

Prandtl, Ludwig. 1933E1. Neuere Ergebnisse der Turbulenzforschung: Recent results of turbulence research. *Zeitschrift des Vereines deutscher Ingenieure*. **7** (5) 105–114. NACA-TM-720. 1933.

Prandtl, Ludwig. 1933E2 Herstellung einwandfreier Luftstrome (Windkanale): Attaining a steady air stream in wind tunnels. *Handbuch der Experimentalphysik*. Vol. 4, (2) 65–106, 1932; NACA-TM-726, 1933.

Prandtl, Ludwig. 1934. Anwendung der turbulenten Reibungsgesetze auf atmosphärische Strömungen. In *Proceedings of the Fourth International Congress of Applied Mechanics, Cambridge*, 238–239. LPGA 3, 1098–1099.

Prandtl, Ludwig. 1938. Beitrag zum Turbulenzsymposium. In *Proceedings of the Fifth International Congress on Applied Mechanics, Cambridge MA*, J.P. Den Hartog and H. Peters (eds), John Wiley, New York, 340–346. LPGA 2, 856–868.

Prandtl, Ludwig. 1942a. *Führer durch die Strömungslehre*. Vieweg, Braunschweig.

Prandtl, Ludwig. 1942b. Zur turbulenten Strömung in Rohren und längs Platten. *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, **4** 18–29. LPGA 2, 632–648.

Prandtl, Ludwig. 1942c. Bemerkungen zur Theorie der Freien Turbulenz. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* **22** 241–243. LPGA 2, 869–873.

Prandtl, Ludwig. 1945. Über die Rolle der Zähigkeit im Mechanismus der ausgebildete Turbulenz: The role of viscosity in the mechanism of developed turbulence. *GOAR* 3712, DLR Archive.

Prandtl, Ludwig. 1948a. Turbulence. *FIAT Review of German Science 1939–1946: Hydro- and Aero-dynamics*, Albert Betz (ed.), Office of Military Government for Germany Field Information Agency Technical, 55–78.

Prandtl, Ludwig. 1948b. Mein Weg zu den Hydrodynamischen Theorien. *Physikalische Blätter*, **3** 89–92. LPGA 3, 1604–1608.

Prandtl, Ludwig. 1948c. *Führer durch die Strömungslehre*. Vieweg, Braunschweig.

Prandtl, Ludwig. 1949E. Bericht über Untersuchungen zur ausgebildeten Turbulenz: Report on investigation of developed turbulence. NACA-TM-1231, 1949.

Prandtl, Ludwig and Eisner, Franz. 1932. Nachtrag zum 'Reibungswiderstand'. In *Hydromechanische Probleme des Schiffsantriebs. Hamburg*, G. Kempf, E. Foerster (eds), 407.

Prandtl, Ludwig et al. 1932. Erörterungsbeiträge. *Hydromechanische Probleme des Schiffsantriebs. Hamburg*, G. Kempf, E. Foerster (eds), 87–98.

Prandtl, Ludwig and Schlichting, Hermann. 1934. Das Widerstandsgesetz rauher Platten. *Werft, Reederei, Hafen*, **21** 1–4. LPGA 2, 649–662.

Prandtl, Ludwig and Reichardt, Hans. 1934. Einfluss von Wärmeschichtung auf Eigenschaften einer turbulenten Strömung. *Deutsche Forschung*, **15** 110–121. LPGA 2, 846–855.

Prandtl, Ludwig and Wieghardt, Karl. 1945. Über ein neues Formelsystem für die ausgebildete Turbulenz. *Nachrichten der Akademie der Wissenschaften zu Göttingen, Mathematisch–Physikalische Klasse*, 6–19.LPGA 2, 874–887.

Rayleigh, Lord. 1887. On the stability or instability of certain fluid motions II. *Proceedings of the London Mathematical Society*, **19** 67–74. Reprinted in *Scientific Papers by John William Strutt, Baron Rayleigh, vol. III (1887–1892)*. Cambridge University Press, Cambridge. 17–23.

Reichardt, H. 1933. Die quadratischen Mittelwerte der Längsschwankungen in der turbulenten Kanalströmung. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **3** 177–180.

Reichardt, H. 1934. Berichte aus den einzelnen Instituten. Physikalisch-Chemisch-Technische Institute. *Naturwissenschaften*, **22** 351.

Reichardt, H. 1935. Die Torsionswaage als Mikromanometer. *Zschr. f. Instrumentenkunde*, **55** 23–33.

Reichardt, H. 1948E. The torsion balance as a micromanometer. NRC-TT-84, 1948-11-13.

Reichardt, H. 1938a. Vorträge aus dem Gebiet der Aero- und Hydrodynamik. Über das Messen turbulenter Längs- und Querschwankungen. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **18** 358–361.

Reichardt, H. 1938b. Messungen turbulenter Spannungen. *Naturwissenschaften*, **26** 404–408.

Reichardt, H. 1941. Über die Theorie der freien Turbulenz. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **21** 257–264.

Reichardt, H. 1942. Gesetzmässigkeiten der freien Turbulenz. *VDI Forschungsheft*, **414**, 22 pages.

Reichardt, H. 1944. Impuls- und Wärmeaustausch in freier Turbulenz. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **24** 268–272.

Reichardt, H. 1951E. On the recording of turbulent longitudinal and transverse fluctuations. NACA-TM-1313, 1951.

Rotta, Julius C. 1990. *Die Aerodynamische Versuchsanstalt in Göttingen, ein Werk Ludwig Prandtls. Ihre Geschichte von den Anfängen bis 1925*. Vandenhoeck und Ruprecht, Göttingen.

Rotta, Julius C. 2000. Ludwig Prandtl und die Turbulenz. In *Ludwig Prandtl, ein Führer in der Strömungslehre: Biographische Artikel zum Werk Ludwig Prandtls*, Gerd E.A. Meier (ed), 53–123.

Sachse, Carola and Walker, Mark (eds). 2005. *Politics and Science in Wartime: Comparative International Perspectives on Kaiser Wilhelm Institutes*. University of Chicago Press.

Saffman, P.G. 1992. *Vortex Dynamics*. Cambridge University Press, Cambridge.

Schiller, Ludwig. 1921. Experimentelle Untersuchungen zum Turbulenzproblem. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, **1** 436–444.

Schlichting, Hermann. 1933. Zur Entstehung der Turbulenz bei der Plattenströmung. *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen*, 181–208.

Schlichting, Hermann. 1949. Lecture series "boundary layer theory", part ii: Turbulent flows. *NACA-TM-1218*, 1949. Translation of "Vortragsreihe" W.S. 1941/42, Luftfahrtforschungsanstalt Hermann Göring, Braunschweig.

Schmaltz, Florian. 2005. *Kampfstoff-Forschung im Nationalsozialismus. Zur Kooperation von Kaiser-Wilhelm-Instituten, Militär und Industrie*. Wallstein, Göttingen.

Schuh, H. 1945. Die Messungen sehr kleiner Windschwankungen (Windkanalturbulenz). *Untersuchungen und Mitteilungen der Deutschen Luftfahrtforschung*, **6623**.

Schuh, H. 1946. *Windschwankungsmessungen mit Hitzdrähten*. AVA Monographien, D1, Chapter 4.3.

Schultz-Grunow, Fritz. 1940. Neues Reibungswiderstandsgesetz für glatte Platten. *Luftfahrtforschung*, **17** 239–246.

Schultz-Grunow, Fritz. 1941E. New frictional resistance law for smooth plates. *NACA-TM-986*, 1941.

Simmons, L.F.G., Salter, C. and Taylor, G.I. 1938. An experimental determination of the spectrum of turbulence. *Proc. Roy Soc. Lond. A*, **165** 73–89.

Sommerfeld, Arnold. 1935. Zu L. Prandtls 60. Geburtstag am 4. Februar 1935. *ZAMM*, **15** 1–2.

Spalding, D.B. 1991. Kolmogorov's two-equation model of turbulence. *Proc. Roy. Soc. Math. Phys. Sci.*, **434** 211–216.

Tani, Itiro. 1977. History of boundary-layer theory. *Ann. Rev. Fluid Mech.*, **9** 87–111.

Tietjens, Oskar. 1925. Beiträge zur Entstehung der Turbulenz. *ZAMM*, **5** 200–217.

Taylor, G.I. 1929. Stability of a viscous liquid contained between two rotating cylinders. *Phys. Trans. Roy. Soc.*, **223** 289–343.

Taylor, G.I. 1935a. Turbulence in a contracting stream. *ZAMM*, **15** 91–96.

Taylor, G.I. 1935b. Statistical theory of turbulence. *Proc. Roy. Soc. Lond. A* **151** 421–444.

Taylor, G.I. 1935c. Statistical theory of turbulence II. *Proc. Roy. Soc. Lond. A* **151** 444–454.

Taylor, G.I. 1935d. Statistical theory of turbulence. III. Distribution of dissipation of energy in a pipe over its cross-section. *Proc. Roy. Soc. Lond. A* **151** 455–464.

Taylor, G.I. 1935e. Statistical theory of turbulence. IV. Diffusion in a turbulent air stream. *Proc. Roy. Soc. Lond. A* **151** 465–478.

Taylor, G.I. 1936. Correlation measurements in a turbulent flow through a pipe. *Proc. Roy. Soc. Lond. A* **157** 537–546.

Taylor, G.I. 1938a. The spectrum of turbulence. *Proc. Roy. Soc. Lond. A* **164** 476–490.

Taylor, G.I. 1938b. Some recent developments in the study of turbulence. In *Proceedings of the Fifth International Congress on Applied Mechanics, Cambridge MA*, J.P. Den Hartog and H. Peters (eds), John Wiley, New York, 294–310.

Tollmien, Walter. 1926. Berechnung turbulenter Ausbreitungsvorgänge. *ZAMM*, **6** 468–478.

Tollmien, Walter. 1929. Über die Entstehung der Turbulenz. *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen*, 21–44.

Trischler, Helmuth. 1994. Self-mobilization or resistance? Aeronautical research and National Socialism. In *Science, Technology and National Socialism*, Monika Renneberg and Mark Walter (eds), Cambridge University Press, Cambridge, 72–87.

Vogel-Prandtl, Johanna. 1993. *Ludwig Prandtl. Ein Lebensbild. Erinnerungen. Dokumente*. Max-Planck-Institut für Strömungsforschung, Göttingen, 1993. (– Mitteilungen aus dem MPI für Strömungsforschung, Nr. 107).

Weizsäcker, von, C.F. 1948. Das Spektrum der Turbulenz bei grossen Reynoldsschen Zahlen. *Zeitschrift für Physik*, **124** 614–627.

Wieghardt, Karl. 1941. Zusammenfassender Bericht über Arbeiten zur statistischen Turbulenztheorie. *Luftfahrt-Forschung*, FB 1563.

Wieghardt, Karl. 1942. Erhöhung des turbulenten Reibungswiderstandes durch Oberflächenstörungen. *ZWB*, FB 1563.

Wieghardt, Karl. 1942E. Correlation of data on the statistical theory of turbulence. NACA-TM-1008, 1942.

Wieghardt, Karl. 1943. Über die Wandschubspannung in turbulenten Reibungsschichten bei veränderlichem Aussendruck. *ZWB*, UM 6603.

Wieghardt, Karl. 1944. Zum Reibungswiderstand rauher Platten. *ZWB*, UM 6612.

Wieghardt, Karl. 1947. *Der Rauhigkeitskanal des Kaiser Wilhelm-Instituts für Strömungsforschung in Göttingen*. AVA-Monographien D1 3.3.

Wieghardt, Karl and Tillmann, W. 1944. Zur turbulenten Reibungsschicht bei Druckanstieg. *ZWB*, UM 6617.

Wieghardt, Karl and Tillmann, W. 1951E. On the turbulent friction layer for rising pressure. NACA-TM-1314, 1951.

Wieselsberger, Carl. 1914. Der Luftwiderstand von Kugeln. *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, **5** 140–145.