Environmental Fluid Dynamics: Lecture 5

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Overview

1. Atmospheric Thermodynamics
    - Hypsometric Equation
    - Reducing Pressure to Seal Level
    - 1st Law of Thermodynamics
    - Adiabatic Processes

Atmospheric Thermodynamics

Atmospheric Thermodynamics: Hypsometric Equation

- Recall from last class that the geopotential thickness between two pressure levels is

$$Z_2 - Z_1 = \frac{R_d}{g_0} \int_{p_2}^{p_1} T_v \frac{dp}{p}$$

- If we assume the atmosphere is **isothermal** and neglect the virtual temperature correction

$$Z_2 - Z_1 = H \ln\left(\frac{p_1}{p_2}\right)$$

where

$$H \equiv \frac{RT}{g_0} = 29.3T$$

is the **scale height**

Atmospheric Thermodynamics: Hypsometric Equation

- Scale height

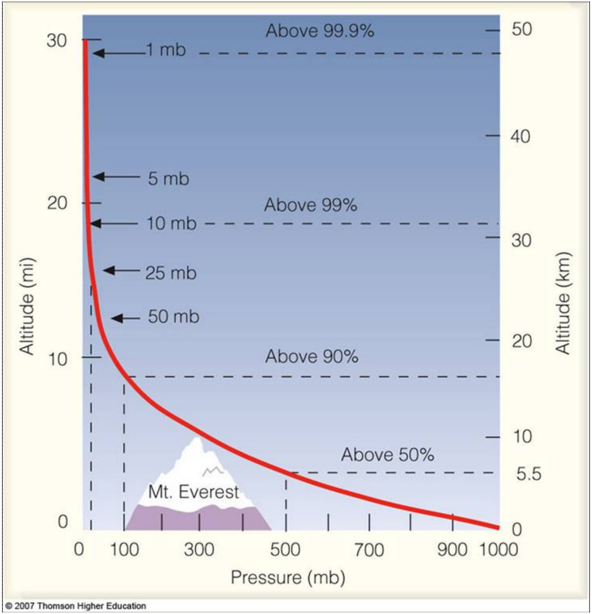
> *The height within which some parameter, such as pressure or density, decreases by a factor 1/e in an isothermal atmosphere*

- We can see this by rearranging the thickness equation

$$Z_2 - Z_1 = H \ln\left(\frac{p_1}{p_2}\right) \Rightarrow p_2 = p_1 \exp\left[-\frac{(Z_2 - Z_1)}{H}\right]$$

- So, H may be thought of as a measure of the effective "thickness" of an atmospheric layer

Atmospheric Thermodynamics: Hypsometric Equation



Atmospheric Thermodynamics: Hypsometric Equation

- As we discussed earlier, temperature usually will vary with height
- It is not always appropriate to neglect the virtual temperature correction
- A more general approach is to integrate T_v w.r.t. presure as

$$\bar{T}_v \equiv \frac{\int_{p2}^{p_1} T_v d(\ln p)}{\int_{p2}^{p_1} d(\ln p)} = \frac{\int_{p2}^{p_1} T_v \dfrac{dp}{p}}{\ln\left(\dfrac{p_1}{p_2}\right)}$$

- Which leads to the **hypsometric equation**

$$Z_2 - Z_1 = \bar{H} \ln\left(\frac{p_1}{p_2}\right) = \frac{R_d \bar{T}_v}{g_0} \ln\left(\frac{p_1}{p_2}\right)$$

Atmospheric Thermodynamics: Reducing P to SLP

- In areas with mountainous terrain, changes in surface pressure between one site and another are largely caused by differences in elevation
- We reduce surface pressure to a common level in order to separate the portion of the pressure field caused by weather
- For a layer between the Earth's surface and sea level:

$$Z_g = \overline{H} \ln \left(\frac{p_0}{p_g} \right)$$

- We can use this to solve for sea-level pressure (SLP)

$$p_0 = p_g \exp \left(\frac{Z_g}{\overline{H}} \right) = p_g \exp \left(\frac{g_0 Z_g}{R_d \overline{T}_v} \right)$$

Atmospheric Thermodynamics: Reducing P to SLP

$$p_g \exp\left(\frac{g_0 Z_g}{R_d \overline{T}_v}\right)$$

- if $Z_g/\overline{H} \ll 1$, then $\exp\left(Z_g/\overline{H}\right) \sim 1 + Z_g/\overline{H}$ and we can rewrite as
$$p_0 - p_g \simeq p_g \frac{Z_g}{\overline{H}} = p_g \left(\frac{g_0 Z_g}{R_d \overline{T}_v}\right)$$
- If we use representative values of $p_g \simeq 1000\ \mathrm{hPa}$ and $\overline{H} \simeq 8000\ \mathrm{m}$, then the pressure correction is

$$p_0 - p_g \sim \frac{Z_g}{8}$$

This means that pressure decreases by $1\ \mathrm{hPa}$ every $8\ \mathrm{m}$ of ascent (within first couple hundred meters above/below SLP)

Atmospheric Thermodynamics: 1$^{\text{st}}$ Law of Thermo

- Imagine a closed system of unit mass
- Thermal energy Q (J) is added to the system via conduction and/or radiation
- In response, the system may do some amount of external work W
- The excess energy given to body over the external work done by the body is given by $Q - W$

Atmospheric Thermodynamics: 1$^{\text{st}}$ Law of Thermo

- It follows from conservation of energy that the internal energy of the system must increase by $Q - W$, or

  $$dQ - dW = dU$$

  where
  - $dQ \rightarrow$ differential heat added to system
  - $dW \rightarrow$ differential work done by system
  - $dU \rightarrow$ differential increase in internal energy
- This describes4 the **First Law of Thermodynamics**
- The change in internal energy only depends on the initial and final states of the system - and not the transfer mechanism

Atmospheric Thermodynamics: 1^{st} Law of Thermo

Joule's Law

The internal energy of a fixed mass of an ideal gas depends only on its temperature (not pressure or volume)

- Found that if a gas expands without doing external work and without taking/giving heat, then the temperature does not change
- This is only possible if molecules of an ideal gas do not exert forces on each other

Atmospheric Thermodynamics: 1^{st} Law of Thermo

- Imagine heat dQ is given to a unit mass, which causes the temperature to increase from T to $T + dT$
- $dQ/dT \rightarrow$ **specific heat**
- If volume is held constant, then

$$c_v = \left(\frac{dQ}{dT}\right)_{\text{v const}}$$

- Also if volume is constant, $dW = 0$, which means $dQ = dU$:

$$c_v = \left(\frac{dU}{dT}\right)_{\text{v const}}$$

- Invoking Joule's Law means U only depends on T, so

$$c_v = \left(\frac{dU}{dT}\right)$$

Atmospheric Thermodynamics: 1st Law of Thermo

- The First Law of Thermodynamics for an ideal gas

$$dQ = c_v dT + p d\alpha$$

  where $\alpha = 1/\rho$ is specific volume
- Adding heat will either change T or α
- The change in U is given by

$$dU = \int_{T_1}^{T_2} c_v dT$$

Atmospheric Thermodynamics: 1^{st} Law of Thermo

- We can also define a specific heat at *constant pressure*

$$c_p = \left(\frac{dQ}{dT}\right)_{\text{p const}}$$

- Here, heat is added, temperature rises, and the system expands - but the pressure remains constant
- Some amount of the heat added to the system is expended to expand against constant pressure of environment
- Thus, more heat must be added to raise a material's temperature by a given amount than if volume had been kept constant

Atmospheric Thermodynamics: 1st Law of Thermo

$$dQ = c_v dT + p d\alpha$$

- Using product rule, $d(p\alpha) = p d\alpha + \alpha dp$, so

$$dQ = c_v dT + d(p\alpha) - \alpha dp$$

- Recall ideal gas law

$$p\alpha = RT$$

so

$$d(p\alpha) = d(RT) = RdT + \cancelto{0}{TdR} = RdT$$

- We can rewrite as

$$dQ = c_v dT + RdT - \alpha dp = (c_v + R)dT - \alpha dp$$

Atmospheric Thermodynamics: 1^{st} Law of Thermo

$$dQ = (c_v + R)dT - \alpha dp$$

- At constant pressure, $\alpha dp = 0$, so

$$\begin{aligned} dQ &= (c_v + R)dT \\ \left(\frac{dQ}{dT}\right)_{\text{p const}} &= c_v + R \\ &\boxed{c_p = c_v + R} \end{aligned}$$

Atmospheric Thermodynamics: 1st Law of Thermo

- For dry air, $c_v = 717 \text{ J K}^{-1} \text{ kg}^{-1}$ and $c_p = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$
- Note that $1004 \text{ J K}^{-1} \text{ kg}^{-1} - 717 \text{ J K}^{-1} \text{ kg}^{-1} = 287 \text{ J K}^{-1} \text{ kg}^{-1}$, which is the gas constant for dry air R_d
- Using $c_p = c_v + R$ and $dQ = (c_v + R)dT$, we can rewrite the First Law of Thermodynamics as

$$dQ = c_p dT - \alpha dp$$

- So, in terms of specific heat, we have

$$dQ = c_v dT + p d\alpha$$
$$dQ = c_p dT - \alpha dp$$

Atmospheric Thermodynamics: 1^{st} Law of Thermo

- Imagine heat is added to a material at constant p such that α increases
- The work done my a unit mass of the material is $p(\alpha_2 - \alpha_1)$
- Thus, the finite heat added to a unit mass at constant pressure is

$$\Delta Q = (U_2 - U_1) + p(\alpha_2 - \alpha_1) = (u_2 + p\alpha_2) - (u_1 + p\alpha_1)$$

- We can rewrite this as

$$\Delta Q = H_2 - H_1$$

where $H \equiv U + p\alpha$ is the **enthalpy**

Atmospheric Thermodynamics: 1st Law of Thermo

$$H \equiv U + p\alpha$$

- Integrating our expression for enthalpy gives

$$dH = dU + d(p\alpha)$$

- Recall that $c_v = (dU/dT)$ and $dQ = c_v dT + d(p\alpha) - \alpha dp$, so $dU = c_v dT = dQ - d(p\alpha) + \alpha dp$
- We can combine

$$\begin{aligned} dH &= dU + d(p\alpha) \\ &= dQ - d(p\alpha) + \alpha dp + d(p\alpha) \\ &= dQ + \alpha dp \end{aligned}$$

which gives another form of the First Law of Thermodynamics

$$dQ = dH - \alpha dp$$

Atmospheric Thermodynamics: 1st Law of Thermo

$$dQ = dH - \alpha dp$$

- Recall, $dQ = c_p dT - \alpha dp$
- Comparing with the enthalpy version above, we see that

  $$dH = c_p dT$$

  or if integrated

  $$H = c_p T$$

  where $h = 0$ when $T = 0$
- Thus, H is the heat required to raise the temperature of a material from $0\ \mathrm{K}$ to $T\ \mathrm{K}$ at constant pressure

Atmospheric Thermodynamics: 1st Law of Thermo

- Imagine some slice of air is at rest and in hydrostatic balance
- If that slice is heated (radiative transfer), then the weight of the overlying air remains constant
- *i.e.*, the heating is at constant pressure
- The increased energy added to the air appears in the form of an increase in enthalpy
- In atmospheric science, enthalpy is referred to as *sensible heat*

Atmospheric Thermodynamics: 1^{st} Law of Thermo

- In the case of heating at constant pressure ($\alpha dp = 0$), we have

$$dQ = dH = c_p dT$$

- This slice of air expands as it is heated, which results in work by pushing up the overlying air against gravity
- Recall $dQ = c_v dT + p d\alpha$, so for the energy given to the air, $c_v dT$ is the increase in internal energy and $p d\alpha = R dT$ is the work done on the overlying air

Atmospheric Thermodynamics: Adiabatic Processes

Adiabatic Process

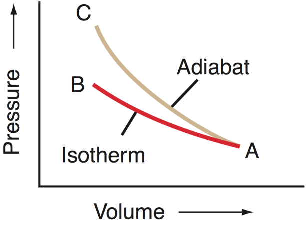
When a material undergoes a physical state change (pressure, volume, temperature) without any heat exchange

- So, $dQ = 0$ for an adiabatic process
- Thus, for $dQ = c_v dT + p d\alpha$, we have

$$-c_v dT = p d\alpha$$

- This means that expansion (compression) requires $dT < 0$ ($dT > 0$) or reduction (increase) in internal energy

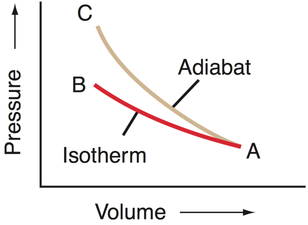
Atmospheric Thermodynamics: Adiabatic Processes



From Wallace and Hobbs (2006)

- Consider initial state at point A
- An isothermal change is AB
- A similar change in volume under adiabatic conditions is AC (called an adiabat)

Atmospheric Thermodynamics: Adiabatic Processes



From Wallace and Hobbs (2006)

- Why is AC steeper?
- Recall $dQ = 0$, and we have $c_v dT = -p d\alpha$
- For compression, $p d\alpha < 0$, so $dT > 0$
- Conversely, for AB the temperature remains constant
- Thus, $T_C > T_B$ and $p_C > p_B$

Atmospheric Thermodynamics: Adiabatic Processes

- Mixing is often viewed as a result the random motions of individual molecules for many fluid dynamics applications
- However, molecular mixing in the atmosphere is only relevant in the lowest cm or at $> 105\ \mathrm{km}$
- In between, vertical mixing is generally is accomplished by larger scale “air parcels”
- We will use *parcel theory* to try and better understand vertical mixing in the atmosphere

Atmospheric Thermodynamics: Adiabatic Processes

Assumptions for an air parcel of infinitesimal dimension

- Thermally insulated - temperature changes adiabatically as it moves vertically
- Thus, the parcel pressure remains equal to the environmental pressure - which is assumed to be in hydrostatic balance
- Moves slowly enough such that macroscopic kinetic energy is a negligible fraction of its total energy

So, if we lift an air parcel adiabatically (no transfer of energy across its surface) and bring it back to its original location then the pressure of the parcel will be the same - a **reversible process**

In the real world these conditions are likely violated due to radiation and condensation

Atmospheric Thermodynamics: Adiabatic Processes

Dry Adiabatic Lapse Rate

The rate of change of temperature with height of a parcel of dry air that satisfies the assumptions of adiabatic mixing

- Recall $dQ = c_p dT - \alpha dp$
- Since $dQ = 0$, we get $c_p dT = \alpha dp$
- This leads to $dT/dp = \alpha/c_p = 1/(\rho c_p)$
- Making use of the hydrostatic equation $dp = -\rho g dz$

$$-\frac{dT}{\rho g dz} = \frac{1}{\rho c_p}$$

- Rearranging yields

$$\frac{dT}{dz} = -\frac{g}{c_p}$$

Atmospheric Thermodynamics: Adiabatic Processes

$$\frac{dT}{dz} = -\boxed{\frac{g}{c_p}}$$

- Let's look at units of g/c_p

$$\frac{[\mathrm{m\ s^{-2}}]}{[\mathrm{J\ K^{-1}\ kg^{-1}}]} = \frac{[\mathrm{m\ s^{-2}}]}{[\mathrm{kg\ m^2\ s^{-2}\ K^{-1}\ kg^{-1}}]} = \frac{[\mathrm{K}]}{[\mathrm{m}]}$$

- It is the **dry adiabatic lapse rate** Γ_d

$$\frac{dT}{dz} = -\Gamma_d$$

Atmospheric Thermodynamics: Adiabatic Processes

$$\frac{dT}{dz} = -\Gamma_d$$

- In the lowest $10\ \mathrm{km}$, g does not change much (see last lecture) and so Γ_d is approximately constant
- At sea level

$$\Gamma_d = \frac{9.81\ \mathrm{m\ s^{-2}}}{1004\ \mathrm{J\ K^{-1}\ kg^{-1}}} = 0.0098\ \mathrm{K\ m^{-1}} = 9.8\ \mathrm{K\ km^{-1}}$$

- Again, this is based on assuming adiabatic lifting/lowering - not something that really happens the atmosphere exactly
- Measurements indicate true lapse rate in the troposphere as

$$\Gamma = \frac{\partial T}{\partial z} \approx 6 - 7\ \mathrm{K\ km^{-1}}$$

Atmospheric Thermodynamics: Adiabatic Processes

Potential Temperature - θ

The temperature that the parcel of air would have if it were expanded or compressed adiabatically from its existing pressure and temperature to a standard pressure p_0 *(generally taken as* $1000\ \mathrm{hPa}$*)*

- A change in pressure results in a temperature change in an adiabatic process
- We must consider this when comparing displaced fluid elements with their surroundings

Atmospheric Thermodynamics: Adiabatic Processes

- We can derive an expression for potential temperature using the First Law of Thermodynamics
- Recall for an adiabatic process, $dQ = 0$, so $c_p dT - \alpha dp = 0$
- Using the ideal gas law, $\alpha = RT/p$, which leads to

$$c_p dT - \frac{RT}{p} dp = 0$$
$$\frac{c_p}{R}\frac{dT}{T} - \frac{dp}{p} = 0$$

- Then we integrate from p_0 (where $T = \theta$) upward

$$\frac{c_p}{R}\int_{\theta}^{T}\frac{dT}{T} = \int_{p_0}^{p}\frac{dp}{p} \Rightarrow \frac{c_p}{R}\ln\left(\frac{T}{\theta}\right) = \ln\left(\frac{p}{p_0}\right)$$

Atmospheric Thermodynamics: Adiabatic Processes

$$\frac{c_p}{R}\ln\left(\frac{T}{\theta}\right) = \ln\left(\frac{p}{p_0}\right)$$

- Take the antilog of both sides

$$\left(\frac{T}{\theta}\right)^{c_p/R} = \frac{p}{p_0}$$

- Rearrange to get potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

This is called **Poisson's equation**

Atmospheric Thermodynamics: Adiabatic Processes

$$\theta = T \left(\frac{p_0}{p} \right)^{R/c_p}$$

- Generally, $R \approx R_d = 287 \text{ J K}^{-1} \text{ kg}^{-1}$ and $c_p = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$ so that $R/c_p = \kappa \simeq 0.286$
- p_0 is usually taken as 1000 hPa - reference pressire
- Potential temperature is a **conserved** quantity because it remains constant for an air parcel as it moves adiabatically
- The atmosphere is approximately adiabatic, so potential temperature is very useful parameter since it remains basically constant - like density in an incompressible fluid

Atmospheric Thermodynamics: Adiabatic Processes

- The potential temperature removes the effect of dry adiabatic temperature changes
- It is valid for
    - ideal gas
    - dry
    - isentropic
    - constant specific heats

Atmospheric Thermodynamics: Adiabatic Processes

- We can show that:

$$\begin{aligned}
\frac{\partial \theta}{\partial z} &\simeq \frac{\partial T}{\partial z} + \Gamma \\
\Delta\theta &\simeq \Delta T + \Gamma \Delta z \\
\theta - \theta_0 &= T - T_0 + \Gamma_d (z - z_0) \\
\theta(z) &= T(z) + \Gamma_d z \qquad (\text{if } z_0 = 0)
\end{aligned}$$

Atmospheric Thermodynamics: Adiabatic Processes

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa}$$

$$\frac{\partial}{\partial z}(\ln\theta = \ln T + \kappa \ln p_0 - \kappa \ln p)$$

$$\frac{1}{\theta}\frac{\partial \theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{\kappa}{p}\frac{\partial p}{\partial z}$$ we treated $\kappa \ln p_0$ as constant

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Using hydrostatic approximation and ideal gas law

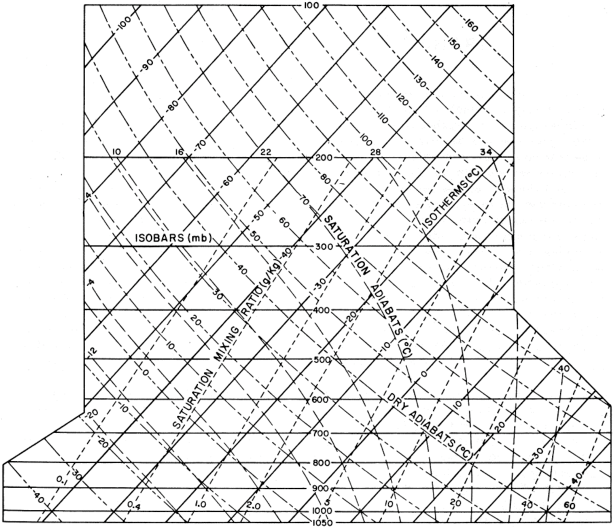
$$-\frac{\kappa}{p}\frac{\partial p}{\partial z} = \frac{\kappa}{p}\rho g = \frac{\kappa \rho g}{\rho R T} = \frac{\kappa g}{RT} = \frac{Rg}{c_p R T} = \frac{1}{T}\frac{g}{c_p} = \frac{1}{T}\Gamma_d$$

$$\frac{\partial \theta}{\partial z} = \frac{\theta}{T}\left(\frac{\partial T}{\partial z} + \Gamma_d\right)$$

$$\frac{\partial \theta}{\partial z} \simeq \left(\frac{\partial T}{\partial z} + \Gamma_d\right)$$ (assume $\theta/T \approx 1$)

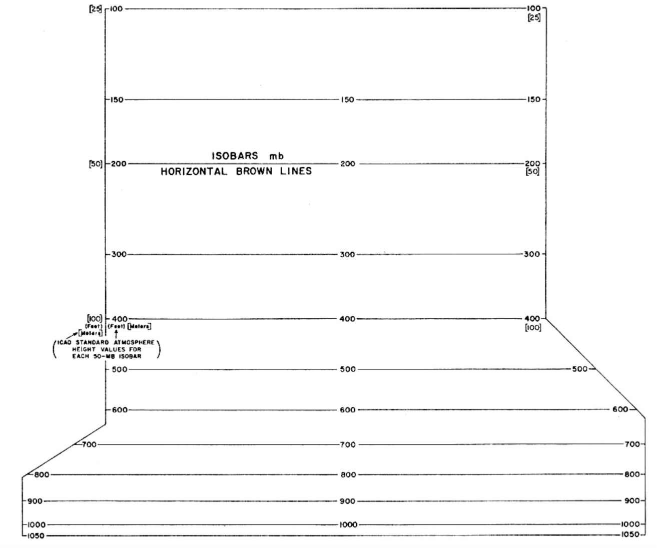
Atmospheric Thermodynamics: Adiabatic Processes

- Visualize these relationships using skew T-ln p diagram



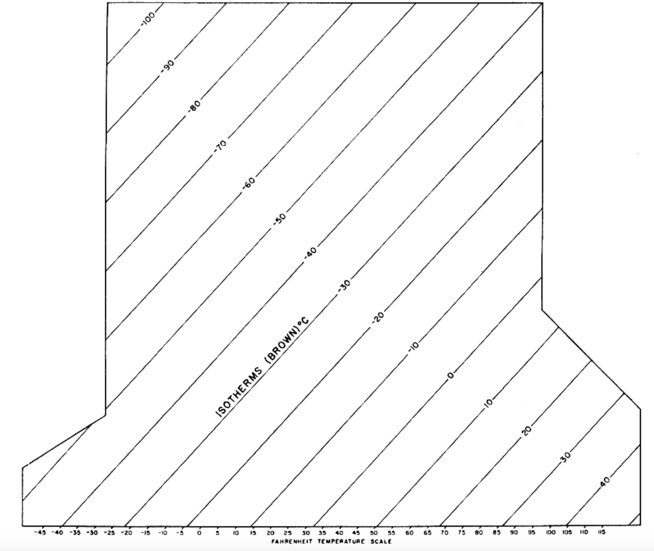
Atmospheric Thermodynamics: Adiabatic Processes

- Isobars



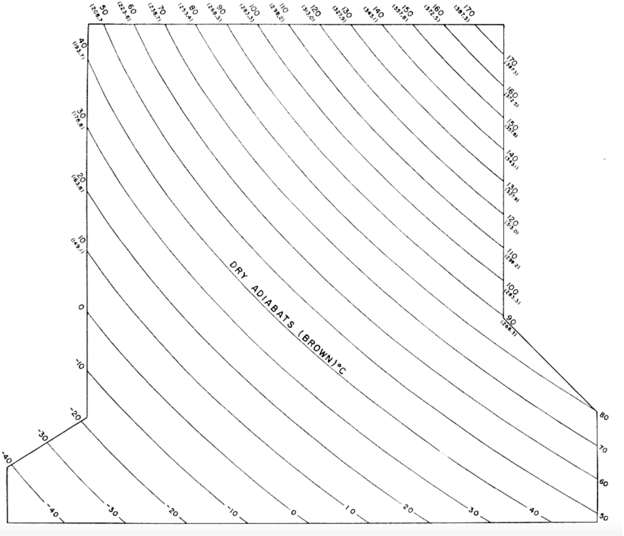
Atmospheric Thermodynamics: Adiabatic Processes

- Isotherms



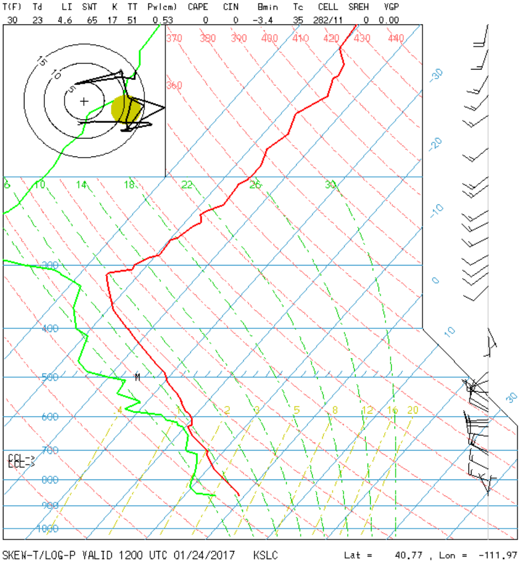
Atmospheric Thermodynamics: Adiabatic Processes

- Dry Adiabats



Atmospheric Thermodynamics: Adiabatic Processes

- Today's atmospheric sounding



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