3

The statistical description of turbulent flows

3.1 The random nature of turbulence

In a turbulent flow, the velocity field $\boldsymbol{U}(\boldsymbol{x}, t)$ is *random*. What does this statement mean? Why is it so?

As a first step we need to understand the word 'random.' Consider a fluid-flow experiment that can be repeated many times under a specified set of conditions, $\mathcal{C}$, and consider an event A, such as $A \equiv \{U < 10 \text{ m s}^{-1}\}$, where U is a specified component of velocity at a specified position and time (measured from the initiation of the experiment). If the event A inevitably occurs, then A is *certain* or *sure*. If the event A cannot occur, then it is *impossible*. The third possibility is that A may occur or it may but need not occur. In this case the event A is *random*. Then, in the example $A \equiv \{U < 10 \text{ m s}^{-1}\}$, U is a *random variable*.

A mistake that is sometimes made is to attribute incorrectly additional significance to the designation 'random,' and then to dispute the fact that turbulence is a random phenomenon. That the event A is random means only that it is neither certain nor impossible. That U is a random variable means only that it does not have a unique value – the same every time the experiment is repeated under the same set of conditions, $\mathcal{C}$. Figure 3.1 illustrates the values $U^{(n)}(n = 1, 2, \ldots, 40)$ taken by the random variable U on 40 repetitions of the experiment.

The next issue to resolve is the consistency between the random nature of turbulent flows, and the deterministic nature of classical mechanics embodied in the Navier–Stokes equations. If the equations of motion are deterministic, why are the solutions random? The answer lies in the combination of two observations.

(i) In any turbulent flow there are, unavoidably, perturbations in initial conditions, boundary conditions, and material properties.

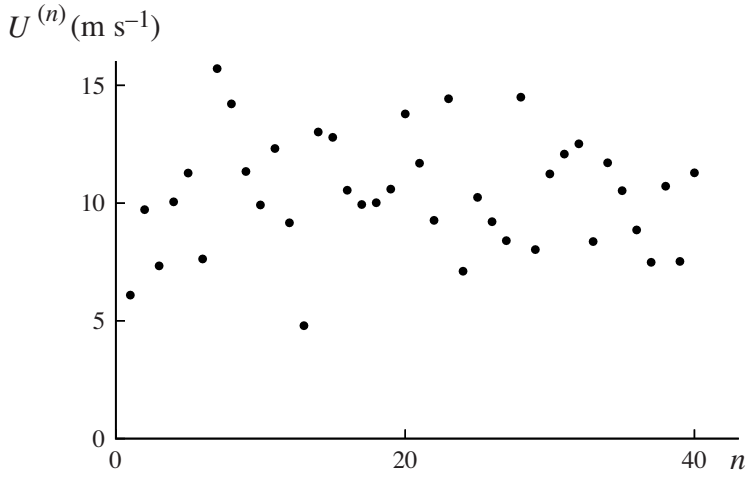

Fig. 3.1. A sketch of the value $U^{(n)}$ of the random velocity variable U on the nth repetition of a turbulent-flow experiment.

(ii) Turbulent flow fields display an acute sensitivity to such perturbations.

At the outset of our discussion on randomness, we considered 'a fluid-flow experiment that can be repeated many times *under a specified set of conditions* $\mathcal{C}$.' An example is the flow of pure water at 20 °C through a smooth straight pipe. It should be appreciated that the conditions, $\mathcal{C}$, thus defined are incomplete: in practice there are, inevitably, perturbations from these nominal conditions. There can be perturbations in boundary conditions, for example, through vibration of the apparatus, or from the detailed finish of nominally smooth surfaces. There can be perturbations in fluid properties caused by small inhomogeneities in temperature or by the presence of impurities, and there can be perturbations in the initial state of the flow. With care and effort these perturbations can be reduced, but they cannot be eliminated. Consequently, the nominal conditions $\mathcal{C}$ are incomplete, and hence do not uniquely determine the evolution of the turbulent flow.

The presence of perturbations does not by itself explain the random nature of turbulent flows – for, indeed, such perturbations are also present in laminar flows. However, at the high Reynolds numbers of turbulent flows, the evolution of the flow field is extremely sensitive to small changes in initial conditions, boundary conditions, and material properties. This sensitivity is well understood in the study of dynamical systems, and has been popularized in books on chaos (e.g., Gleick (1988) and Moon (1992)). It is now demonstrated using the Lorenz equations.

Lorenz (1963) studied a time-dependent system, characterized by three state variables, $x(t), y(t)$, and $z(t)$. These variables evolve according to the

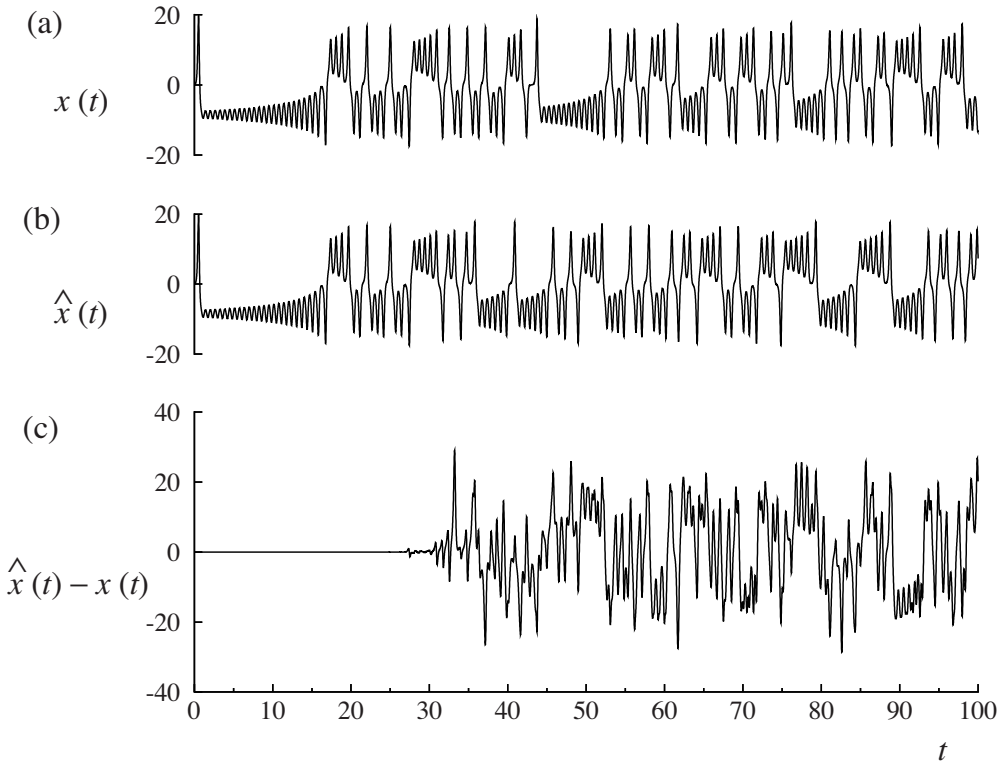

Fig. 3.2. Time histories from the Lorenz equations (Eqs. (3.1)): (a) $x(t)$ from the initial condition Eq. (3.2); (b) $\hat{x}(t)$ from the slightly different initial condition Eq. (3.3); and (c) the difference $\hat{x}(t) - x(t)$.

ordinary differential equations

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{3.1}$$

where the coefficients are $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$. For the initial condition

$$[x(0), y(0), z(0)] = [0.1, 0.1, 0.1], \tag{3.2}$$

Fig. 3.2(a) shows the time history $x(t)$ obtained from the numerical integration of Eqs. (3.1). The result obtained – denoted by $\hat{x}(t)$ – with the slightly different initial condition

$$[x(0), y(0), z(0)] = [0.100\,001, 0.1, 0.1], \tag{3.3}$$

is shown in Fig. 3.2(b). It may be observed that (as expected) $x(t)$ and $\hat{x}(t)$ are initially indistinguishable, but by $t = 35$ they are quite different. This observation is made clearer in Fig. 3.2(c), which shows the difference $\hat{x}(t) - x(t)$.

A consequence of this extreme sensitivity to initial conditions is that – beyond some point – the state of the system cannot be predicted. In this example, if the initial state is known only to within 10^{-6}, then Fig. 3.2 clearly shows that no useful prediction can be made beyond $t = 35$.

This example serves to demonstrate that a simple set of deterministic equations – much simpler than the Navier–Stokes equations – can exhibit acute sensitivity to initial conditions, and hence unpredictability.

The qualitative behavior of the Lorenz system depends on the coefficients. In particular, for the fixed values $\sigma = 10$ and $\beta = \frac{8}{3}$, the behavior depends on ρ. If ρ is less than a critical value $\rho^* \approx 24.74$, then the system goes to a stable fixed point, i.e., the state variables $[x(t), y(t), z(t)]$ tend asymptotically to fixed values. However, for $\rho > \rho^*$ (e.g., $\rho = 28$ as in Fig. 3.2) chaotic behavior ensues. Again, there is a similarity to the Navier–Stokes equations, which (with steady boundary conditions) have steady solutions at sufficiently low Reynolds number, but chaotic, turbulent solutions at high Re. Further discussions of the Lorenz equations, dynamical systems and equations, dynamical systems, and chaos are contained in the books of Guckenheimer and Holmes (1983), Moon (1992), and Gleick (1988).

3.2 Characterization of random variables

For a laminar flow, we can use theory (i.e., the Navier–Stokes equations) to calculate U (a particular component of the velocity at a specified position and time), and we can perform an experiment to measure U. From a century of experience, we have a high degree of confidence that the calculated and measured values of U will agree (to within small numerical and experimental errors).

The Navier–Stokes equations apply equally to turbulent flows, but here the aim of theory must be different. Since U is a random variable, its value is inherently unpredictable: a theory that predicts a particular value for U is almost certain to be wrong. A theory can, however, aim at determining the *probability* of events such as $A \equiv \{U < 10 \text{ m s}^{-1}\}$.

In this section we develop the concepts and tools used to characterize a random variable such as U. In particular U is completely characterized by its *probability density function* (PDF). The random velocity field $\boldsymbol{U}(\boldsymbol{x}, t)$ in a turbulent flow is a much more complicated mathematical object than the single random variable U. In subsequent sections we introduce some quantities used to characterize sets of random variables (e.g., U_1, U_2, and U_3), random functions of time (e.g., $U(t)$), and random functions of position (e.g., $U(\boldsymbol{x})$).

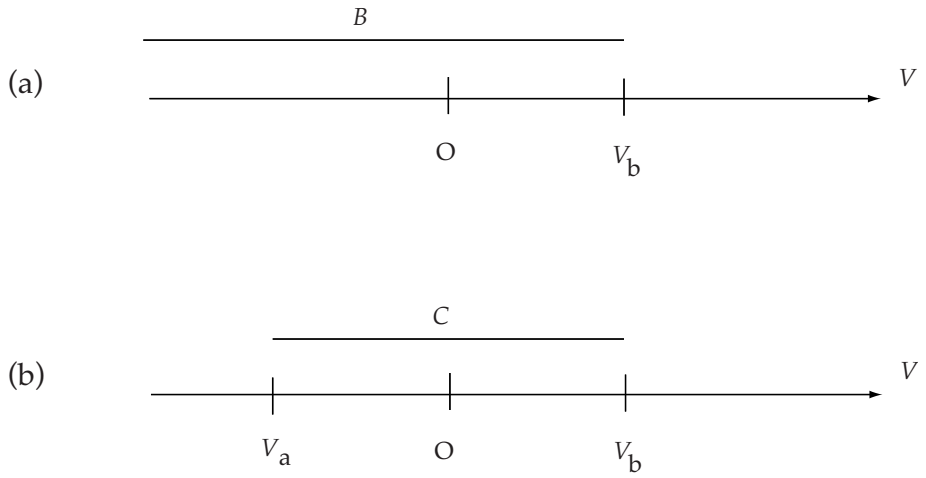

Fig. 3.3. Sketches of the sample space of U showing the regions corresponding to the events (a) $B \equiv \{U < V_b\}$, and (b) $C \equiv \{V_a \leq U < V_b\}$.

Sample space

In order to be able to discuss more general events than $A \equiv \{U < 10 \text{ m s}^{-1}\}$, we introduce an independent velocity variable V, which is referred to as the *sample-space* variable corresponding to U. As illustrated in Fig. 3.3, different events such as

$$B \equiv \{U < V_b\}, \tag{3.4}$$

$$C \equiv \{V_a \leq U < V_b\}, \quad \text{for} \quad V_a < V_b, \tag{3.5}$$

correspond to different regions of the sample space.

Probability

The *probability* of the event B, for example, is written

$$p = P(B) = P\{U < V_b\}. \tag{3.6}$$

For the moment, the reader's intuitive understanding of probability is sufficient: p is a real number ($0 \leq p \leq 1$) signifying the likelihood of the occurrence of the event. For an impossible event p is zero; for a sure event p is unity. (Probability is discussed further in Section 3.8.)

The cumulative distribution function

The probability of any event can be determined from the *cumulative distribution function* (CDF), which is defined by

$$F(V) \equiv P\{U < V\}. \tag{3.7}$$

For example, we have

$$P(B) = P\{U < V_b\} = F(V_b), \tag{3.8}$$

$$\begin{aligned} P(C) = P\{V_a \le U < V_b\} &= P\{U < V_b\} - P\{U < V_a\} \\ &= F(V_b) - F(V_a). \end{aligned} \tag{3.9}$$

The three basic properties of the CDF are

$$F(-\infty) = 0, \tag{3.10}$$

since $\{U < -\infty\}$ is impossible;

$$F(\infty) = 1, \tag{3.11}$$

since $\{U < \infty\}$ is certain; and,

$$F(V_b) \ge F(V_a), \quad \text{for} \quad V_b > V_a, \tag{3.12}$$

since the probability of every event is non-negative, i.e.

$$F(V_b) - F(V_a) = P\{V_a \le U < V_b\} \ge 0. \tag{3.13}$$

The third property (Eq. (3.12)) expresses the fact that the CDF is a non-decreasing function.

The probability density function

The *probability density function* (PDF) is defined to be the derivative of the CDF:

$$f(V) \equiv \frac{dF(V)}{dV}. \tag{3.14}$$

It follows simply from the properties of the CDF that the PDF is non-negative

$$f(V) \ge 0, \tag{3.15}$$

it satisfies the normalization condition

$$\int_{-\infty}^{\infty} f(V)\, dV = 1, \tag{3.16}$$

and $f(-\infty) = f(\infty) = 0$. Further, from Eq. (3.13) it follows that the probability of the random variable being in a particular interval equals the integral

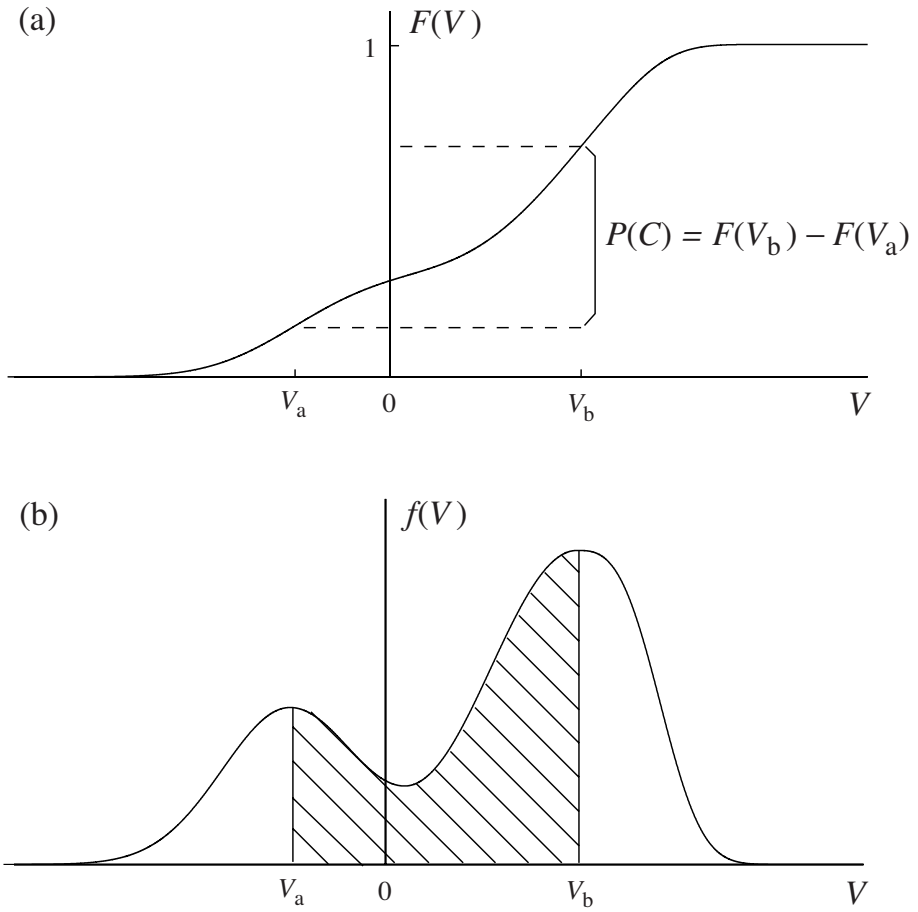

Fig. 3.4. Sketches of (a) the CDF of the random variable U showing the probability of the event $C \equiv \{V_a \leq U < V_b\}$, and (b) the corresponding PDF. The shaded area in (b) is the probability of C.

of the PDF over that interval:

$$\begin{aligned} P\{V_a \leq U < V_b\} &= F(V_b) - F(V_a) \\ &= \int_{V_a}^{V_b} f(V)\,\mathrm{d}V. \end{aligned} \tag{3.17}$$

Figure 3.4 provides a graphical interpretation of this equation.

For an infinitesimal interval, Eq. (3.17) becomes

$$\begin{aligned} P\{V \leq U < V + \mathrm{d}V\} &= F(V + \mathrm{d}V) - F(V) \\ &= f(V)\,\mathrm{d}V. \end{aligned} \tag{3.18}$$

Thus the PDF $f(V)$ is the probability *per unit distance* in the sample space – hence the term 'probability *density* function.' The PDF $f(V)$ has the dimensions of the inverse of U, whereas the CDF and the product $f(V)\,\mathrm{d}V$

are non-dimensional. Under a change of variables, the transformation rule for densities (such as PDF's) is different than that for functions: see Exercise 3.9 on page 49.

It is emphasized that the PDF $f(V)$ (or equally the CDF) fully characterizes the random variable U. Two or more random variables that have the same PDF are said to be *identically distributed*, or equivalently *statistically identical.*

Means and moments

The *mean* (or *expectation*) of the random variable U is defined by

$$\langle U \rangle \equiv \int_{-\infty}^{\infty} V f(V)\, \mathrm{d}V. \tag{3.19}$$

It is the probability-weighted average of all possible values of U. More generally, if $Q(U)$ is a function of U, the mean of $Q(U)$ is

$$\langle Q(U) \rangle \equiv \int_{-\infty}^{\infty} Q(V) f(V)\, \mathrm{d}V. \tag{3.20}$$

Even when the condition is not stated explicitly, it should be understood (here and below) that the mean $\langle Q(U) \rangle$ exists only if the integral in Eq. (3.20) converges absolutely.

The rules for taking means are quite simple. If $Q(U)$ and $R(U)$ are functions of U, and if a and b are constants, then

$$\langle [aQ(U) + bR(U)] \rangle = a\langle Q(U) \rangle + b\langle R(U) \rangle, \tag{3.21}$$

as may readily be verified from Eq. (3.20). Thus the angled brackets $\langle\ \rangle$ behave as a linear operator. While U, $Q(U)$, and $R(U)$ are all random variables, $\langle U \rangle$, $\langle Q(U) \rangle$, and $\langle R(U) \rangle$ are not. Hence the mean of the mean is the mean: $\langle \langle U \rangle \rangle = \langle U \rangle$.

The *fluctuation* in U is defined by

$$u \equiv U - \langle U \rangle, \tag{3.22}$$

and the *variance* is defined to be the mean-square fluctuation:

$$\mathrm{var}(U) \equiv \langle u^2 \rangle = \int_{-\infty}^{\infty} (V - \langle U \rangle)^2 f(V)\, \mathrm{d}V. \tag{3.23}$$

The square-root of the variance is the *standard deviation*

$$\mathrm{sdev}(U) = \sqrt{\mathrm{var}\ (U)} = \langle u^2 \rangle^{1/2}, \tag{3.24}$$

and is also denoted by u' and σ_u, and is also referred to as the r.m.s. (*root mean square*) of U.

The n*th central moment* is defined to be

$$\mu_n \equiv \langle u^n \rangle = \int_{-\infty}^{\infty} (V - \langle U \rangle)^n f(V)\, \mathrm{d}V. \tag{3.25}$$

Evidently we have $\mu_0 = 1$, $\mu_1 = 0$, and $\mu_2 = \sigma_u^2$.

(In contrast, the nth moment about the origin – or the nth raw moment – is defined to be $\langle U^n \rangle$.)

EXERCISES

3.1 With Q and R being random variables, and a and b being constants, use Eq. (3.20) to verify the relations

$$\langle a \rangle = a, \quad \langle aQ \rangle = a\langle Q \rangle, \tag{3.26}$$

$$\langle Q + R \rangle = \langle Q \rangle + \langle R \rangle, \quad \langle \langle Q \rangle \rangle = \langle Q \rangle, \tag{3.27}$$

$$\langle \langle Q \rangle \langle R \rangle \rangle = \langle Q \rangle \langle R \rangle, \quad \langle \langle Q \rangle R \rangle = \langle Q \rangle \langle R \rangle, \tag{3.28}$$

$$\langle q \rangle = 0, \quad \langle q \langle R \rangle \rangle = 0, \tag{3.29}$$

where $q \equiv Q - \langle Q \rangle$.

3.2 Let Q be defined by

$$Q = a + bU, \tag{3.30}$$

where U is a random variable, and a and b are constants. Show that

$$\langle Q \rangle = a + b\langle U \rangle, \tag{3.31}$$

$$\mathrm{var}(Q) = b^2\, \mathrm{var}(U), \tag{3.32}$$

$$\mathrm{sdev}(Q) = b\, \mathrm{sdev}(U). \tag{3.33}$$

Show also that

$$\mathrm{var}(U) = \langle U^2 \rangle - \langle U \rangle^2. \tag{3.34}$$

Standardization

It is often convenient to work in terms of standardized random variables, which, by definition, have zero mean and unit variance. The standardized random variable $\hat{U}$ corresponding to U is

$$\hat{U} \equiv (U - \langle U \rangle)/\sigma_u, \tag{3.35}$$

and its PDF – the standardized PDF of U – is

$$\hat{f}(\hat{V}) = \sigma_u f(\langle U \rangle + \sigma_u \hat{V}). \tag{3.36}$$

The moments of $\hat{U}$ – the standardized moments of U – are

$$\hat{\mu}_n = \frac{\langle u^n \rangle}{\sigma_u^n} = \frac{\mu_n}{\sigma_u^n} = \int_{-\infty}^{\infty} \hat{V}^n \hat{f}(\hat{V})\, \mathrm{d}\hat{V}. \tag{3.37}$$

Evidently we have $\hat{\mu}_0 = 1$, $\hat{\mu}_1 = 0$ and $\hat{\mu}_2 = 1$. The third standardized moment $\hat{\mu}_3$ is called the *skewness*, and the fourth $\hat{\mu}_4$ is the *flatness* or *kurtosis*.

EXERCISE

3.3 Show that the standardized moments of U and Q (defined by Eq. (3.30)) are identical.

The characteristic function

The *characteristic function* of the random variable U is defined by

$$\Psi(s) \equiv \langle e^{iUs} \rangle = \int_{-\infty}^{\infty} f(V) e^{iVs}\, \mathrm{d}V. \tag{3.38}$$

It may be recognized that the integral in Eq. (3.38) is an inverse Fourier transform: $\Psi(s)$ and $f(V)$ form a Fourier-transform pair, and consequently they contain the same information.

The characteristic function is a mathematical device that facilitates some derivations and proofs. Its properties are described in Appendix I. Characteristic functions are used extensively in Chapter 12, but not before. Consequently a study of Appendix I can be deferred.

3.3 Examples of probability distributions

To consolidate the notions developed, and to illustrate some qualitatively different behaviors, we now give some specific examples of probability distributions. These distributions are encountered in later chapters.

The uniform distribution

If U is uniformly distributed in the interval $a \le V < b$, then the PDF of U is

$$f(V) = \begin{cases} \dfrac{1}{b-a}, & \text{for } a \le V < b, \\ 0, & \text{for } V < a \text{ and } V \ge b. \end{cases} \tag{3.39}$$

This PDF and the corresponding CDF are shown in Fig. 3.5.

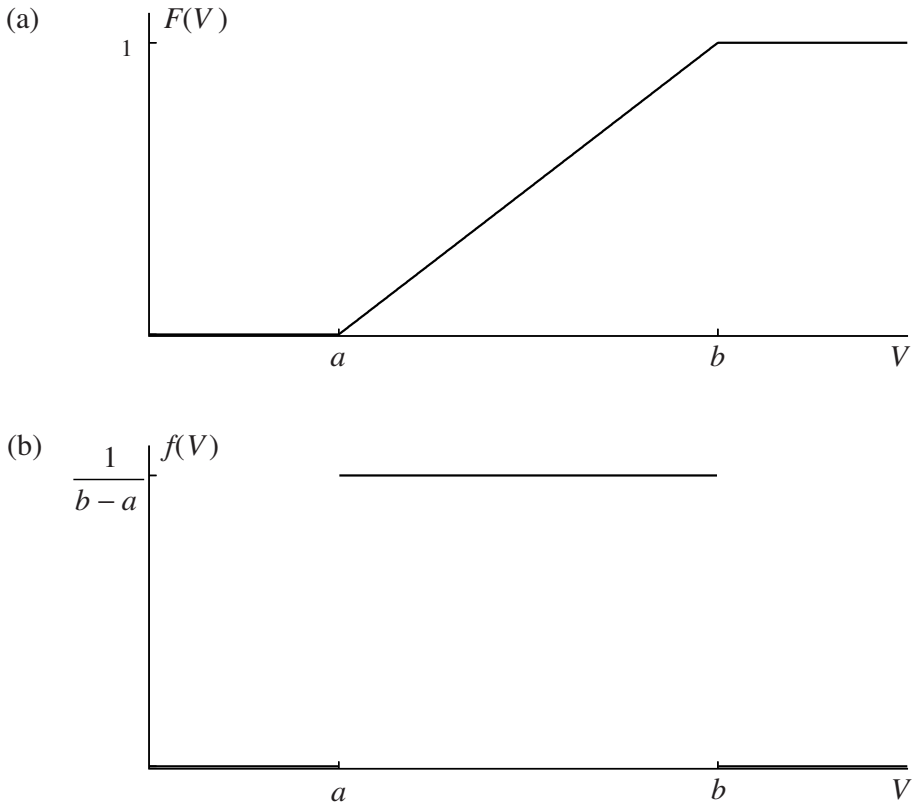

Fig. 3.5. The CDF (a) and the PDF (b) of a uniform random variable (Eq. (3.39)).

EXERCISE

3.4 For the uniform distribution Eq. (3.39) show that

(a) $\langle U \rangle = \frac{1}{2}(a+b)$,
(b) $\mathrm{var}(U) = \frac{1}{12}(b-a)^2$,
(c) $\hat{\mu}_3 = 0$, and
(d) $\hat{\mu}_4 = \frac{9}{5}$.

The exponential distribution

If U is exponentially distributed with parameter λ, then its PDF (see Fig. 3.6) is

$$f(V) = \begin{cases} \dfrac{1}{\lambda}\exp(-V/\lambda), & \text{for } V \geq 0, \\ 0, & \text{for } V < 0. \end{cases} \tag{3.40}$$

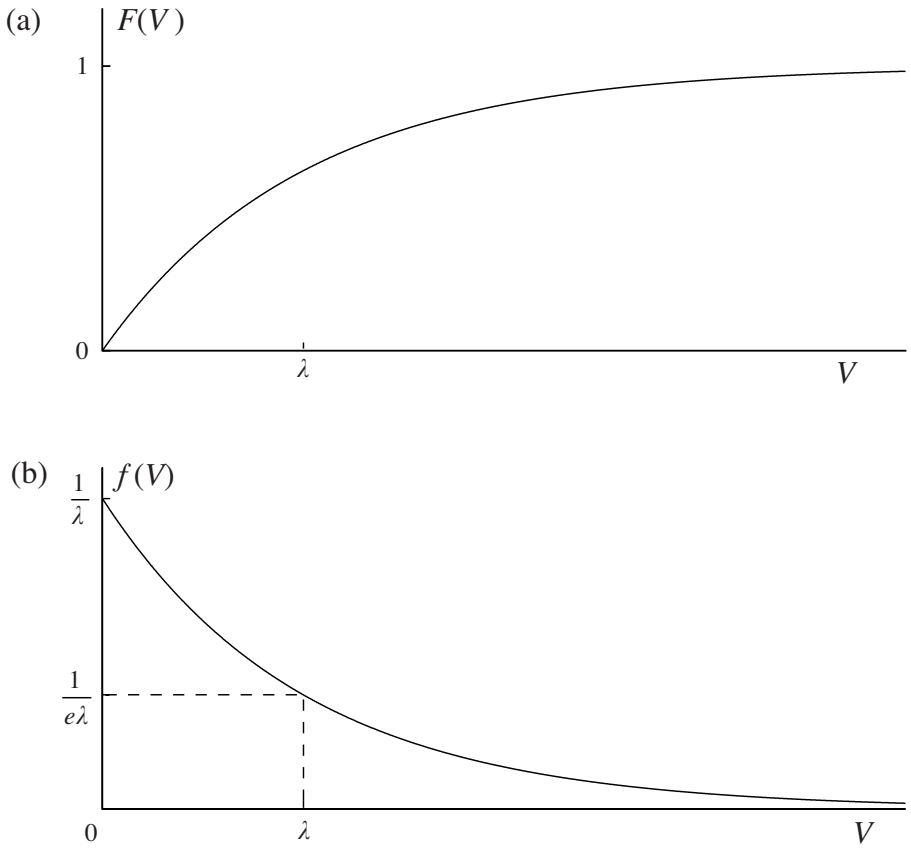

Fig. 3.6. The CDF (a) and PDF (b) of an exponentially distributed random variable (Eq. (3.40)).

EXERCISE

3.5 For the exponential distribution Eq. (3.40) show that

(a) the normalization condition is satisfied,
(b) $\langle U \rangle = \lambda$,
(c) $\langle U^n \rangle = n\lambda \langle U^{n-1} \rangle = n!\lambda^n$, for $n \geq 1$,
(d) $F(V) = 1 - \exp(-V/\lambda)$, for $V > 0$,
$= 0$, for $V \leq 0$, and
(e) $\text{Prob}\{U \geq a\lambda\} = e^{-a}$, for $a \geq 0$.

The normal distribution

Of fundamental importance in probability theory is the normal or *Gaussian* distribution. If U is normally distributed with mean μ and standard deviation σ, then the PDF of U is

$$f(V) = \mathcal{N}(V; \mu, \sigma^2) \equiv \frac{1}{\sigma\sqrt{2\pi}} \exp[-\tfrac{1}{2}(V-\mu)^2/\sigma^2]. \tag{3.41}$$

Here $\mathcal{N}(V;\mu,\sigma^2)$ – or sometimes $\mathcal{N}(\mu,\sigma^2)$ – denotes the normal distribution with mean μ and variance σ^2. We can also write

$$U \overset{\mathrm{D}}{=} \mathcal{N}(\mu,\sigma^2), \tag{3.42}$$

to indicate that U is *equal in distribution* to a normal random variable, i.e., the PDF of U is given by Eq. (3.41).

If U is normally distributed according to Eq. (3.41) then

$$\hat{U} \equiv (U-\mu)/\sigma \tag{3.43}$$

is a *standardized Gaussian random variable* with PDF

$$\hat{f}(V) = \mathcal{N}(V;0,1) = \frac{1}{\sqrt{2\pi}} e^{-V^2/2}. \tag{3.44}$$

This PDF and the corresponding CDF

$$\hat{F}(V) = \int_{-\infty}^{V} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,\mathrm{d}x = \tfrac{1}{2}\left[1 + \operatorname{erf}(V/\sqrt{2})\right] \tag{3.45}$$

are shown in Fig. 3.7.

EXERCISE

3.6 By considering the quantity

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}V}\left(\frac{V^n}{\sqrt{2\pi}} e^{-V^2/2}\right)\mathrm{d}V, \tag{3.46}$$

obtain a recurrence relation for the standardized moments $\hat{\mu}_n$ of the Gaussian distribution. Show that the odd moments ($\hat{\mu}_3$, $\hat{\mu}_5$, …) are zero, that the kurtosis is

$$\hat{\mu}_4 = 3, \tag{3.47}$$

and that the *superskewness* is

$$\hat{\mu}_6 = 15. \tag{3.48}$$

The log-normal distribution

We again take U to be normally distributed with mean μ and variance σ^2. Then the positive random variable

$$Y = e^U \tag{3.49}$$

is, by definition, log-normally distributed.

The CDF $F_Y(y)$ and PDF $f_Y(y)$ of Y can be deduced from those of U,

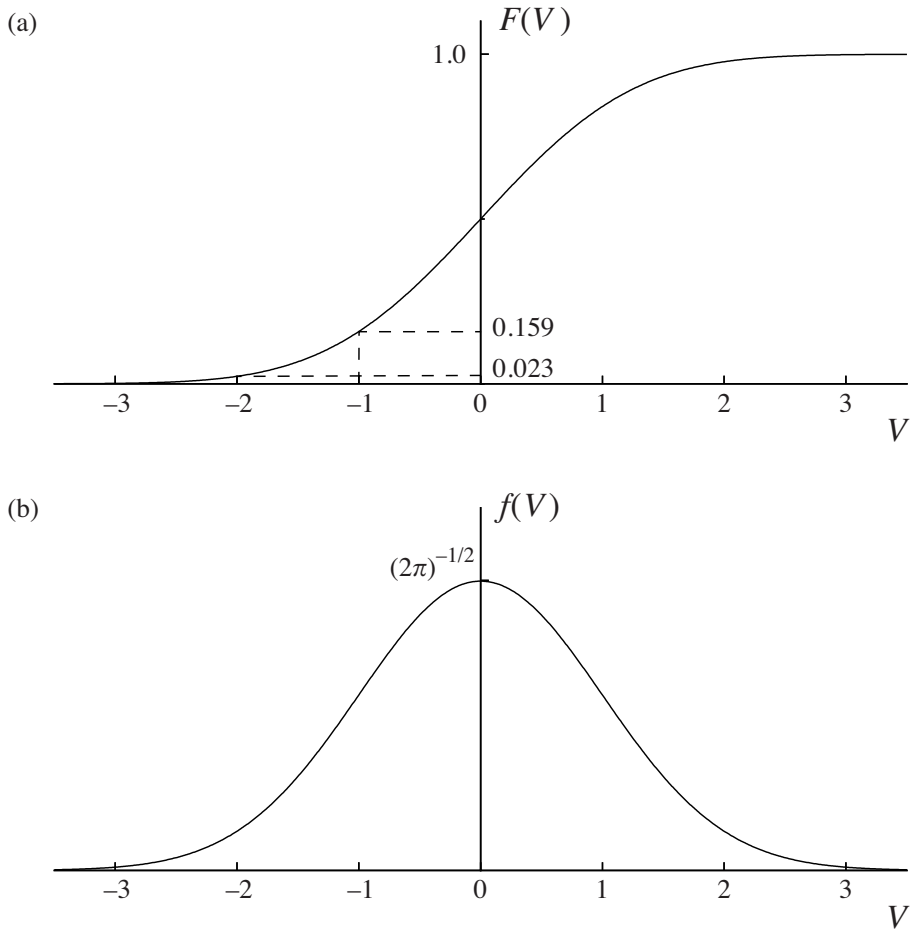

Fig. 3.7. The CDF (a) and PDF (b) of a standardized Gaussian random variable.

namely $F(V)$ and $f(V)$ given by Eq. (3.41). Since Y is positive, the sample space can be taken to be the positive real line, i.e., $y \geq 0$. Starting from the definition of the CDF, we obtain

$$\begin{aligned} F_Y(y) &= P\{Y < y\} = P\{e^U < y\} = P\{U < \ln y\} \\ &= F(\ln y). \end{aligned} \tag{3.50}$$

The PDF is then obtained by differentiating with respect to y:

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{1}{y} f(\ln y) \\ &= \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(\ln y - \mu)^2/\sigma^2\right]. \end{aligned} \tag{3.51}$$

Figure 3.8 shows the PDF $f_Y(y)$ and the CDF $F_Y(y)$ for $\langle Y \rangle = 1$ and various values of the variance. It may be seen that different values of σ^2

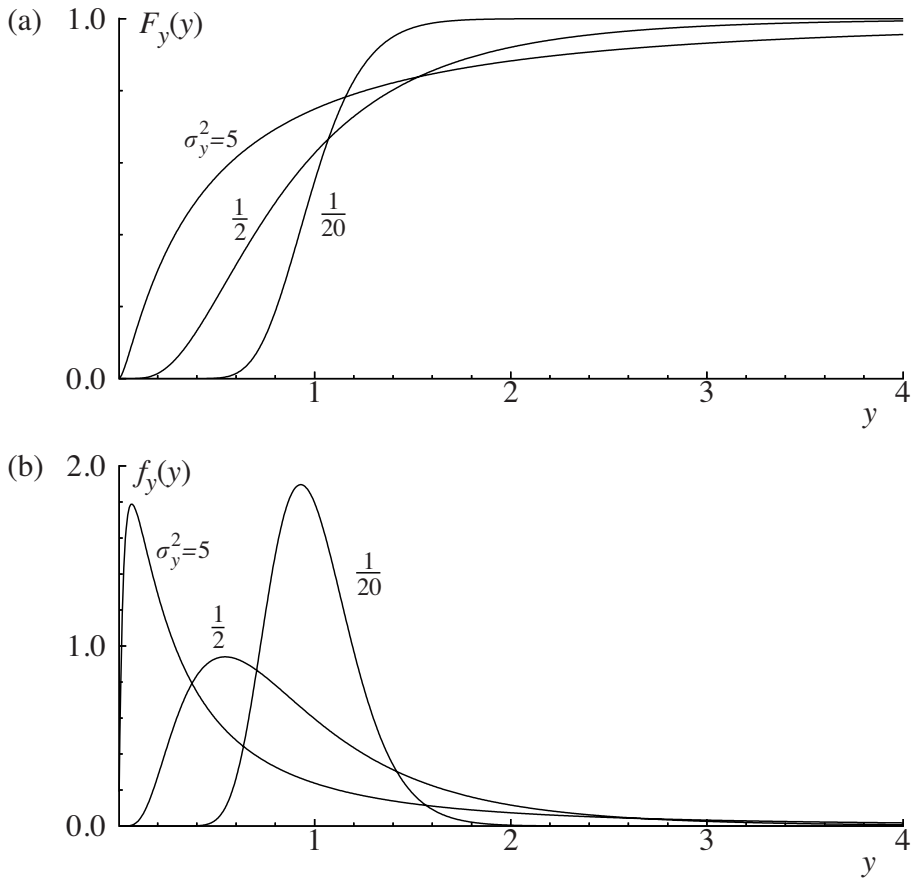

Fig. 3.8. The CDF (a) and PDF (b) of the log-normal random variable Y with $\langle Y \rangle = 1$ and $\text{var}(Y) = \frac{1}{20}$, $\frac{1}{2}$, and 5.

produce different shapes of PDF. In particular, a large value of σ^2 leads to a PDF with a long tail, which is most clearly seen in the CDF's slow approach to unity. As shown in Exercise 3.7, the normalized variance $\text{var}(Y/\langle Y \rangle)$ increases as e^{σ^2}.

Equations (3.50) and (3.51) illustrate the transformation rules for PDFs and CDFs. These are further developed in Exercise 3.9.

EXERCISES

3.7 Show that the raw moments of Y (defined by Eq. (3.49)) are

$$\langle Y^n \rangle = \exp(n\mu + \tfrac{1}{2}n^2\sigma^2). \tag{3.52}$$

(Hint: evaluate $\int_{-\infty}^{\infty} e^{nV} f(V)\,\mathrm{d}V$.)

Show that the specification

$$\mu = -\tfrac{1}{2}\sigma^2 \tag{3.53}$$

results in $\langle Y \rangle$ being unity, and that the variance of Y is

$$\text{var}(Y) = \langle Y \rangle^2 (e^{\sigma^2} - 1). \tag{3.54}$$

3.8 The random variable Z is defined by

$$Z \equiv aY^b, \tag{3.55}$$

where Y is a log-normal random variable, and a and b are positive constants. Show that Z is also log-normal with

$$\text{var}(\ln Z) = b^2 \,\text{var}(\ln Y). \tag{3.56}$$

3.9 The random variable U has the CDF $F(V)$ and PDF $f(V)$. The random variable Y is defined by

$$Y = Q(U), \tag{3.57}$$

where $Q(V)$ is a monotonically increasing function. Following the steps in Eqs. (3.50) and (3.51), show that the CDF $F_Y(y)$ and PDF $f_Y(y)$ for Y are given by

$$F_Y(y) = F(V), \tag{3.58}$$

$$f_Y(y) = f(V) \Big/ \frac{\mathrm{d}Q(V)}{\mathrm{d}V}, \tag{3.59}$$

where

$$y \equiv Q(V). \tag{3.60}$$

Show that the corresponding results for $Q(V)$ being a monotonically decreasing function are

$$F_Y(y) = 1 - F(V), \tag{3.61}$$

$$f_Y(y) = -f(V) \Big/ \frac{\mathrm{d}Q(V)}{\mathrm{d}V}. \tag{3.62}$$

Show that Eqs. (3.59) and (3.62) can be written in the common form

$$f_Y(y)\,\mathrm{d}y = f(V)\,\mathrm{d}V, \tag{3.63}$$

where $\mathrm{d}V$ and

$$\mathrm{d}y \equiv \left| \frac{\mathrm{d}Q(V)}{\mathrm{d}V} \right| \mathrm{d}V \tag{3.64}$$

are corresponding infinitesimal intervals.

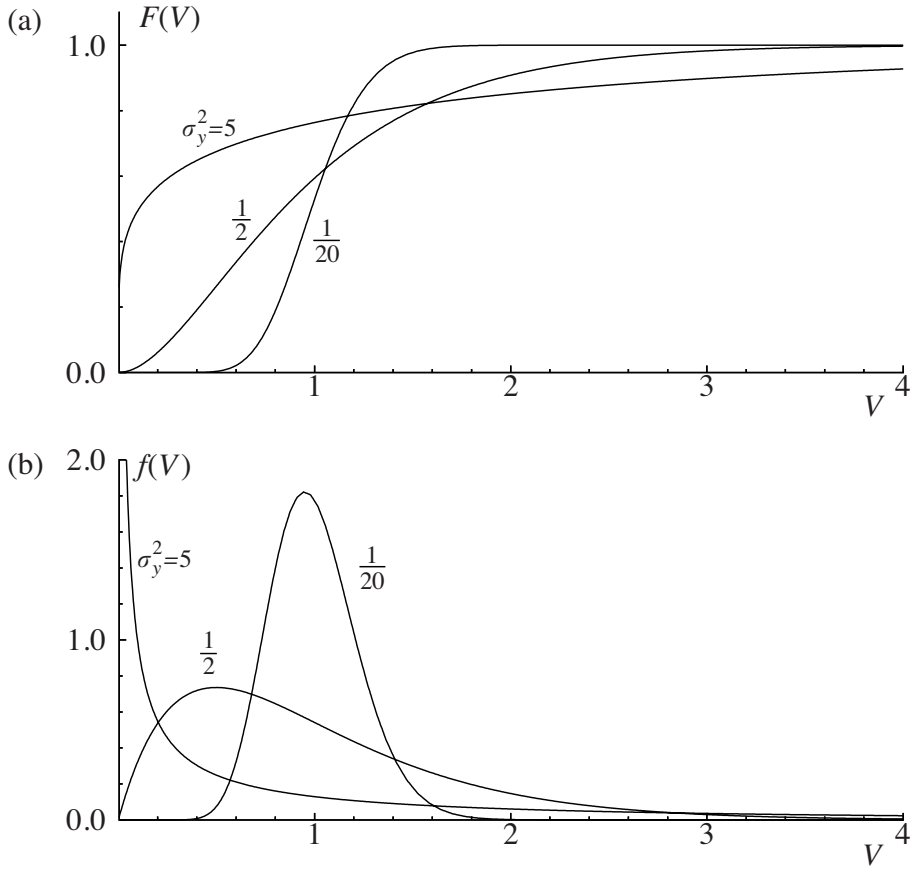

Fig. 3.9. The CDF (a) and PDF (b) for the gamma distribution with mean $\mu = 1$ and variance $\sigma^2 = \frac{1}{20}$, $\frac{1}{2}$, and 5.

The gamma distribution

The positive random variable U, with mean μ and variance σ^2, has a gamma distribution if its PDF is

$$f(V) = \frac{1}{\Gamma(\alpha)} \left(\frac{\alpha}{\mu}\right)^{\alpha} V^{\alpha-1} \exp\left(-\frac{\alpha V}{\mu}\right), \tag{3.65}$$

where α is defined by

$$\alpha \equiv \left(\frac{\mu}{\sigma}\right)^2, \tag{3.66}$$

and $\Gamma(\alpha)$ is the gamma function

$$\Gamma(\alpha) \equiv \int_0^{\infty} x^{\alpha-1} e^{-x}\, \mathrm{d}x. \tag{3.67}$$

For $\alpha = 1$, this becomes the exponential distribution and the value of the PDF at the origin is $f(0) = 1/\mu$. For larger values of α (smaller normalized

variance) the PDF is zero at the origin, whereas for small values of α it is infinite – as is evident in Fig. 3.9.

EXERCISE

3.10 Use the substitution $x = \alpha V/\mu$ to show that the normalized raw moments of the gamma distribution are

$$\int_0^\infty \left(\frac{V}{\mu}\right)^n f(V)\,\mathrm{d}V = \frac{1}{\alpha^n \Gamma(\alpha)} \int_0^\infty x^{n+\alpha-1} e^{-x}\,\mathrm{d}x$$
$$= \frac{\Gamma(n+\alpha)}{\alpha^n \Gamma(\alpha)} = \frac{(n+\alpha-1)!}{\alpha^n (\alpha-1)!}, \tag{3.68}$$

where the last expression applies for integer n and α.
Verify the consistency of this result for $n = 0$, 1, and 2.

Delta-function distributions

Suppose that U is a random variable that takes the value a with probability p, and the value b ($b > a$) with probability $1 - p$. It is straightforward to deduce the CDF of U:

$$F(V) = P\{U < V\} = \begin{cases} 0, & \text{for } V \le a, \\ p, & \text{for } a < V \le b, \\ 1, & \text{for } V > b, \end{cases} \tag{3.69}$$

see Fig. 3.10. This can be written in terms of Heaviside functions as

$$F(V) = pH(V-a) + (1-p)H(V-b). \tag{3.70}$$

The corresponding PDF (obtained by differentiating Eq. (3.70)) is

$$f(V) = p\delta(V-a) + (1-p)\delta(V-b), \tag{3.71}$$

see Fig. 3.10. (The properties of Dirac delta functions and Heaviside functions are reviewed in Appendix C.)

A random variable that can take only a finite number of values is a *discrete random variable* (as opposed to a continuous random variable). Although the tools presented in this section are aimed at describing continuous random variables, evidently (with the aid of Heaviside and Dirac delta functions) discrete random variables can also be treated. Furthermore, if U is a *sure variable*, with probability one of having the value a, its CDF and PDF are consistently given by

$$F(V) = H(V-a), \tag{3.72}$$

$$f(V) = \delta(V-a). \tag{3.73}$$

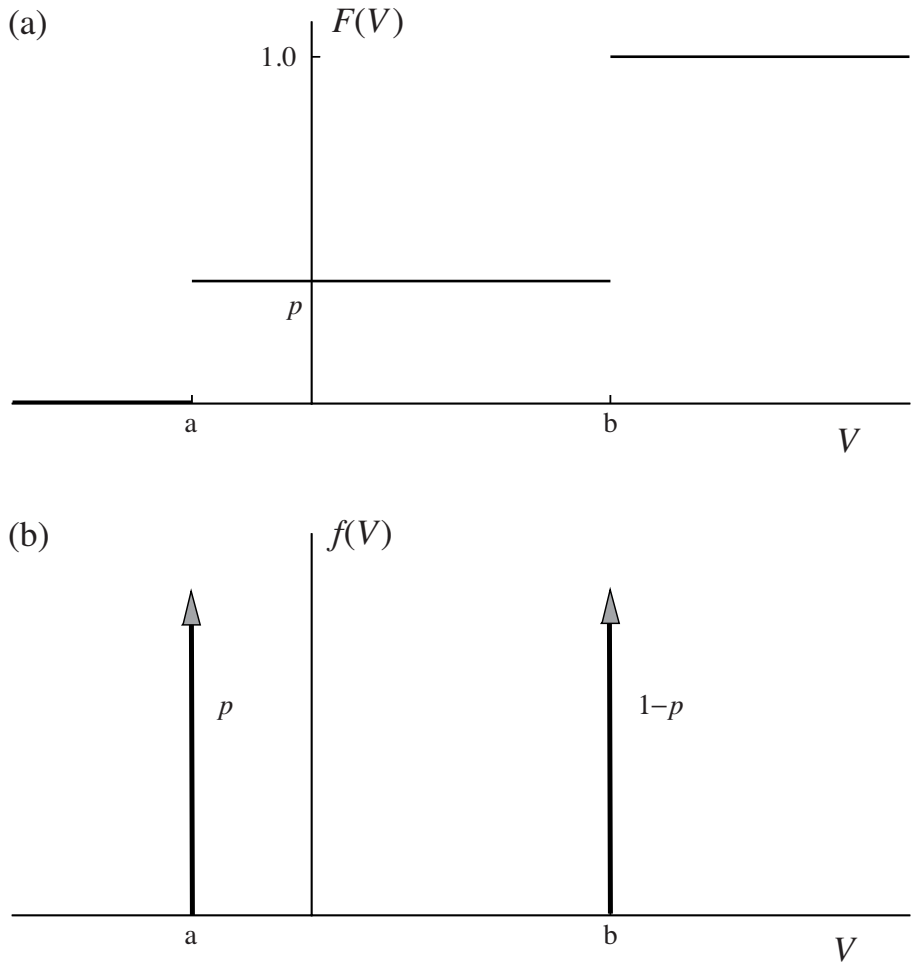

Fig. 3.10. The CDF (a) and the PDF (b) of the discrete random variable U, Eq. (3.69).

EXERCISES

3.11 Let U be the outcome of the toss of a fair die, i.e., $U = 1, 2, 3, 4, 5,$ or 6 with equal probability. Show that the CDF and PDF of U are

$$F(V) = \frac{1}{6}\sum_{n=1}^{6} H(V-n), \tag{3.74}$$

$$f(V) = \frac{1}{6}\sum_{n=1}^{6} \delta(V-n). \tag{3.75}$$

Sketch these distributions.

3.12 Let $f_\phi(\psi)$ be the PDF of a scalar ϕ that satisfies the boundedness condition $\phi_{\min} \le \phi \le \phi_{\max}$. For a given value of the mean $\langle\phi\rangle$, the

maximum possible value of the variance $\langle \phi'^2 \rangle$ occurs when $f_\phi(\psi)$ adopts the double-delta-function distribution

$$f_\phi(\psi) = p\delta(\phi_{\max} - \psi) + (1-p)\delta(\phi_{\min} - \psi). \tag{3.76}$$

For this distribution show that

$$p = \frac{\langle \phi \rangle - \phi_{\min}}{\phi_{\max} - \phi_{\min}}, \tag{3.77}$$

$$\langle \phi'^2 \rangle = (\phi_{\max} - \langle \phi \rangle)(\langle \phi \rangle - \phi_{\min}). \tag{3.78}$$

Note: for $\phi_{\min} = 0$, $\phi_{\max} = 1$, these results are $p = \langle \phi \rangle$ and $\langle \phi'^2 \rangle = \langle \phi \rangle (1 - \langle \phi \rangle)$.

The Cauchy distribution

The mean, variance, and other moments are defined as integrals of the PDF (Eq. (3.20)). We have implicitly assumed that all such integrals converge; and, indeed, with few exceptions, this is true for PDFs encountered in turbulence research. It is useful to have a simple counter-example: this is provided by the Cauchy distribution.

The PDF of the Cauchy distribution centered at c and with half-width w is

$$f(V) = \frac{w/\pi}{(V-c)^2 + w^2}. \tag{3.79}$$

For large V, f varies as V^{-2}, and hence the integral of $Vf(V)$ diverges as $\ln V$. Hence, although the distribution is symmetric about its center $V = c$, nevertheless the mean (defined by Eq. (3.19)) does not exist. The variance is infinite.

Figure 3.11 shows the Cauchy density (Eq. (3.79)) and the corresponding CDF

$$F(V) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{V-c}{w}\right), \tag{3.80}$$

for $c = 0,\ w = 1$.

EXERCISE

3.13 The PDF sketched in Fig. 3.12 has mean zero and unit variance (i.e., it is standardized). Show that the variables defined in the sketch are given by

$$a^2 = \tfrac{6}{11}(1 + 2\sqrt{3}), \quad b = \sqrt{3}\,a, \quad h = \frac{1}{a + \frac{1}{2}b}. \tag{3.81}$$

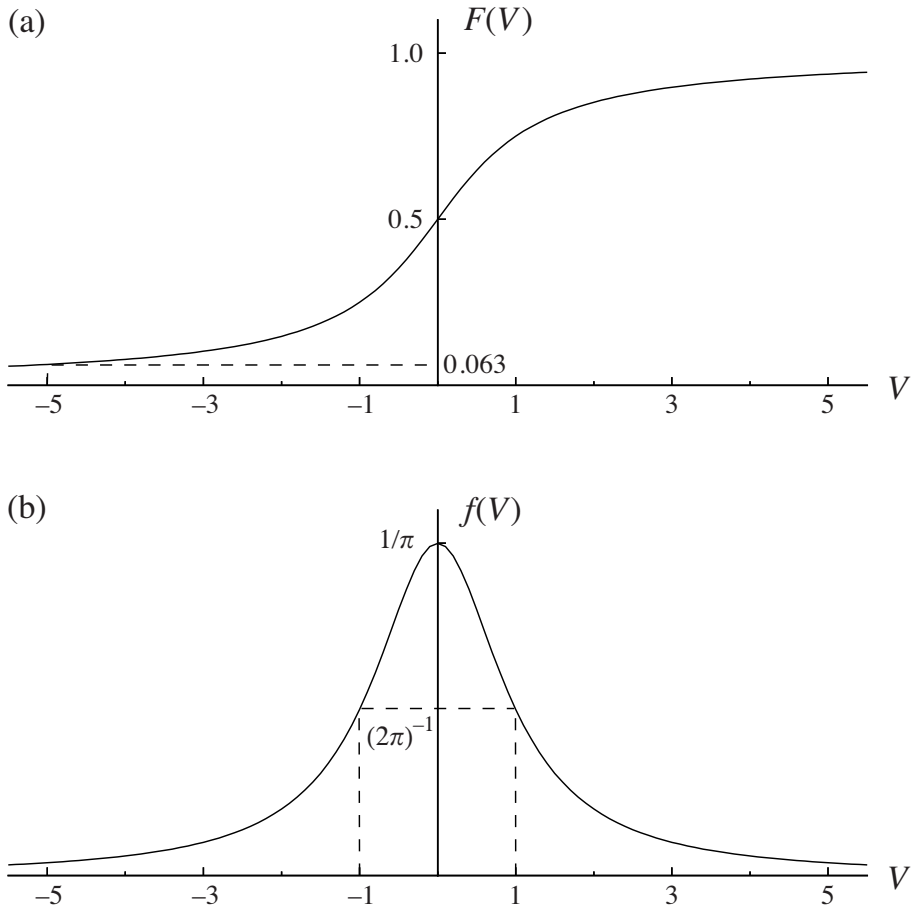

Fig. 3.11. The CDF (a) and PDF (b) for the Cauchy distribution (Eqs. (3.79) and (3.80)) with $c = 0,\ w = 1$.

3.4 Joint random variables

In this section the results obtained for the single random variable U are extended to two or more random variables. We take as an example the components of velocity (U_1, U_2, U_3) at a particular position and time in a turbulent flow.

The sample-space variables corresponding to the random variables $\boldsymbol{U} = \{U_1, U_2, U_3\}$ are denoted by $\boldsymbol{V} = \{V_1, V_2, V_3\}$. For the two components U_1 and U_2, Fig. 3.13 shows a *scatter plot* consisting in the $N = 100$ points $(V_1, V_2) = (U_1^{(n)}, U_2^{(n)}), n = 1, 2, \ldots, N$, where $(U_1^{(n)}, U_2^{(n)})$ are the values of (U_1, U_2) on the nth repetition of the experiment. The CDF of the joint random variables $(U_1,\ U_2)$ is defined by

$$F_{12}(V_1, V_2) \equiv P\{U_1 < V_1, U_2 < V_2\}. \tag{3.82}$$

It is the probability of the sample point $(V_1, V_2) = (U_1, U_2)$ lying within the

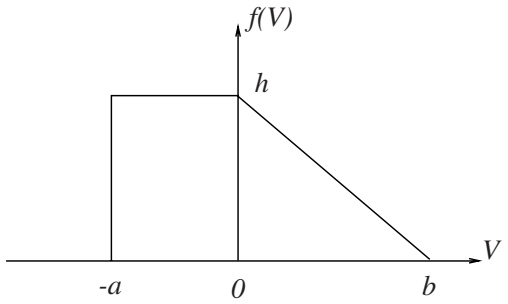

Fig. 3.12. A sketch of the standardized PDF in Exercise 3.13.

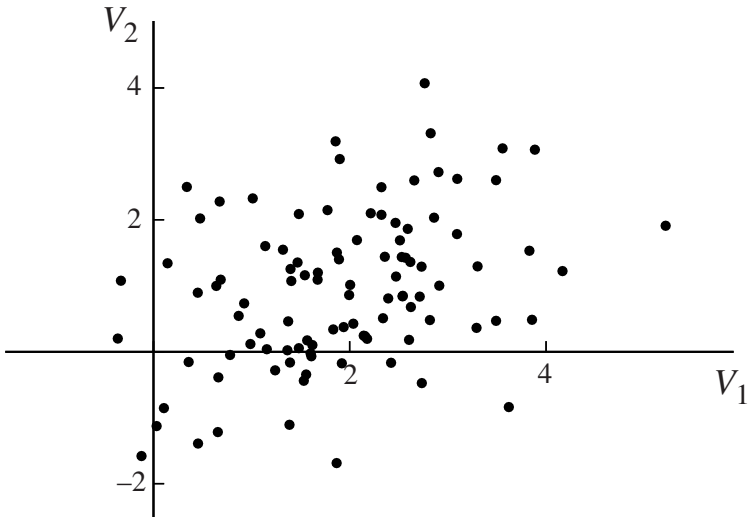

Fig. 3.13. A scatter plot in the V_1–V_2 sample space of 100 samples of the joint random variables (U_1, U_2). (In this example U_1 and U_2 are jointly normal with $\langle U_1 \rangle = 2$, $\langle U_2 \rangle = 1$, $\langle u_1^2 \rangle = 1$, $\langle u_2^2 \rangle = \frac{5}{16}$, and $\rho_{12} = 1/\sqrt{5}$.)

shaded area of Fig. 3.14. Clearly, $F_{12}(V_1, V_2)$ is a non-decreasing function of each of its arguments:

$$F_{12}(V_1 + \delta V_1, V_2 + \delta V_2) \geq F_{12}(V_1, V_2), \quad \text{for all } \delta V_1 \geq 0 \text{ and } \delta V_2 \geq 0. \tag{3.83}$$

Other properties of the CDF are

$$F_{12}(-\infty, V_2) = P\{U_1 < -\infty,\ U_2 < V_2\} = 0, \tag{3.84}$$

since $\{U_1 < -\infty\}$ is impossible; and

$$\begin{aligned} F_{12}(\infty, V_2) &= P\{U_1 < \infty,\ U_2 < V_2\} \\ &= P\{U_2 < V_2\} = F_2(V_2), \end{aligned} \tag{3.85}$$

since $\{U_1 < \infty\}$ is certain. The CDF $F_2(V_2)$ of the single random variable

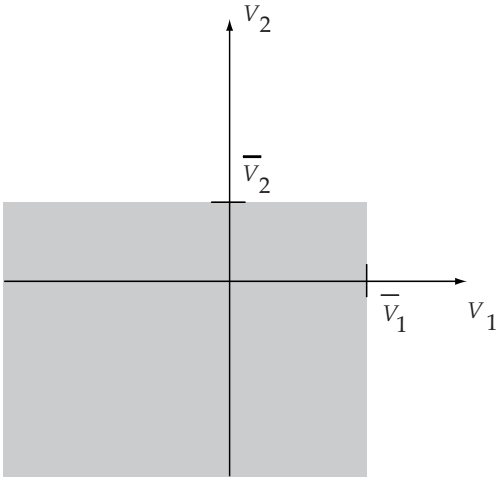

Fig. 3.14. The V_1–V_2 sample space showing the region corresponding to the event $\{U_1 < \overline{V}_1, U_2 < \overline{V}_2\}$.

U_2 (defined in Eq. (3.85)) is called the *marginal CDF*. Similarly, the marginal CDF of U_1 is $F_1(V_1) = F_{12}(V_1, \infty)$.

The joint PDF (JPDF) of U_1 and U_2 is defined by

$$f_{12}(V_1, V_2) \equiv \frac{\partial^2}{\partial V_1 \, \partial V_2} F_{12}(V_1, V_2). \tag{3.86}$$

Its fundamental property, illustrated in Fig. 3.15, is

$$P\{V_{1a} \le U_1 < V_{1b},\ V_{2a} \le U_2 \le V_{2b}\} = \int_{V_{1a}}^{V_{1b}} \int_{V_{2a}}^{V_{2b}} f_{12}(V_1, V_2)\, \mathrm{d}V_2\, \mathrm{d}V_1. \tag{3.87}$$

Other properties, that can readily be deduced, are

$$f_{12}(V_1, V_2) \ge 0, \tag{3.88}$$

$$\int_{-\infty}^{\infty} f_{12}(V_1, V_2)\, \mathrm{d}V_1 = f_2(V_2), \tag{3.89}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{12}(V_1, V_2)\, \mathrm{d}V_1\, \mathrm{d}V_2 = 1, \tag{3.90}$$

where $f_2(V_2)$ is the *marginal* PDF of U_2.

If $Q(U_1, U_2)$ is a function of the random variables, its mean is defined by

$$\langle Q(U_1, U_2) \rangle \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(V_1, V_2) f_{12}(V_1, V_2)\, \mathrm{d}V_1\, \mathrm{d}V_2. \tag{3.91}$$

The means $\langle U_1 \rangle$ and $\langle U_2 \rangle$, and the variances $\langle u_1^2 \rangle$ and $\langle u_2^2 \rangle$, can be determined from this equation, or equally, from the marginal PDFs $f_1(V_1)$ and $f_2(V_2)$ (see Exercise 3.15). Here u_1 and u_2 are the fluctuations, e.g., $u_1 \equiv U_1 - \langle U_1 \rangle$.

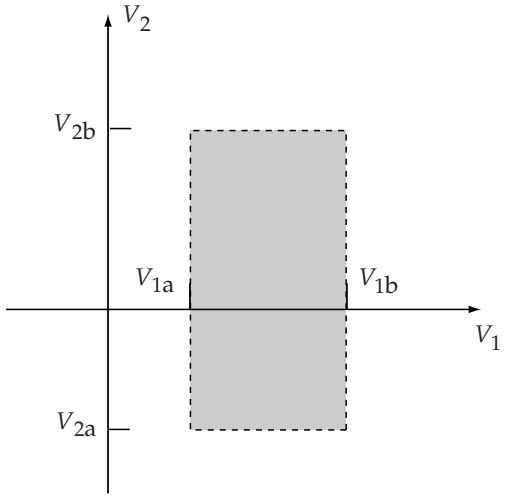

Fig. 3.15. The V_1–V_2 sample space showing the region corresponding to the event $\{V_{1a} \le U_1 < V_{1b},\ V_{2a} \le U_2 < V_{2b}\}$, see Eq. (3.87).

The *covariance* of U_1 and U_2 is the mixed second moment

$$\mathrm{cov}(U_1, U_2) = \langle u_1 u_2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (V_1 - \langle U_1 \rangle)(V_2 - \langle U_2 \rangle) f_{12}(V_1, V_2)\, \mathrm{d}V_1\, \mathrm{d}V_2, \tag{3.92}$$

and the *correlation coefficient* is

$$\rho_{12} \equiv \langle u_1 u_2 \rangle / [\langle u_1^2 \rangle \langle u_2^2 \rangle]^{1/2}. \tag{3.93}$$

As illustrated by the scatter plot in Fig. 3.13, a positive correlation coefficient arises when positive excursions from the mean for one random variable (e.g., $u_1 > 0$) are preferentially associated with positive excursions for the other (i.e., $u_2 > 0$). Conversely, if positive excursions for u_1 are preferentially associated with negative excursions of u_2, as in Fig. 3.16, then the correlation coefficient is negative. In general, we have the *Cauchy–Schwarz inequality*

$$-1 \le \rho_{12} \le 1, \tag{3.94}$$

see Exercise 3.16.

If the correlation coefficient ρ_{12} is zero (which implies that the covariance $\langle u_1 u_2 \rangle$ is zero) then the random variables U_1 and U_2 are *uncorrelated*. In contrast, if ρ_{12} is unity, U_1 and U_2 are *perfectly correlated*; and, if ρ_{12} equals -1, they are *perfectly negatively correlated*. Examples of these correlations are given in Exercise 3.17.

For the scatter plot shown in Fig. 3.16, it is clear that the samples with $U_1 \approx V_{1a}$ and those with $U_1 \approx V_{1b}$ are likely to have significantly different values of U_2. This is confirmed in Fig. 3.17, which shows $f_{12}(V_1, V_2)$ for

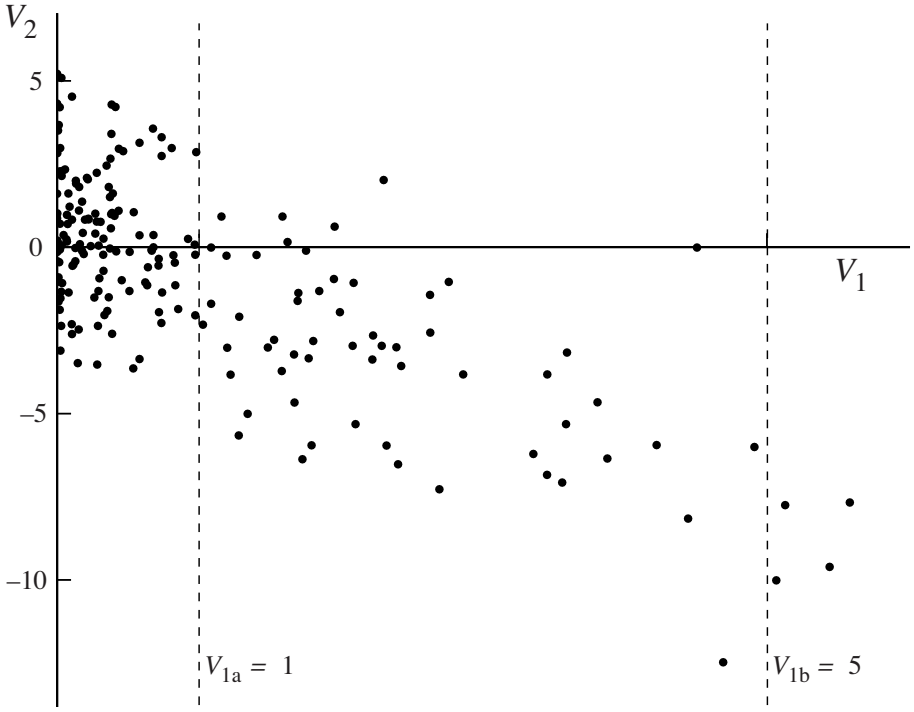

Fig. 3.16. A scatter plot of negatively correlated random variables ($\langle U_1 \rangle = 1$, $\langle U_2 \rangle = -1$, $\langle u_1^2 \rangle = 2$, $\langle u_2^2 \rangle = 12$, and $\rho_{12} = -\sqrt{2/3}$).

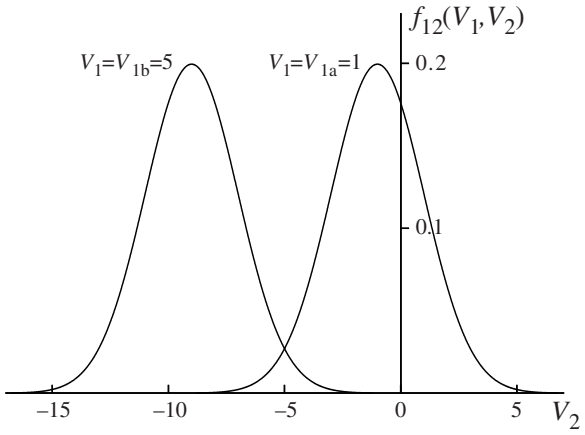

Fig. 3.17. The joint PDF of the distribution shown in Fig. 3.16, plotted against V_2 for $V_1 = V_{1a} = 1$ and $V_1 = V_{1b} = 5$.

$V_1 = V_{1a}$ and $V_1 = V_{1b}$. For fixed V_{1a}, $f_{12}(V_{1a}, V_2)$ indicates how U_2 is distributed for samples (U_1, U_2) with $U_1 = V_{1a}$. These ideas are made precise by defining *conditional* PDFs: the PDF of U_2 conditional on $U_1 = V_1$ is

$$f_{2|1}(V_2|V_1) \equiv f_{12}(V_1, V_2)/f_1(V_1). \tag{3.95}$$

This is simply the joint PDF f_{12}, scaled so that it satisfies the normalization condition

$$\int_{-\infty}^{\infty} f_{2|1}(V_2|V_1)\,\mathrm{d}V_2 = 1. \tag{3.96}$$

For given V_1, if $f_1(V_1)$ is zero, then $f_{2|1}(V_2|V_1)$ is undefined. Otherwise it is readily verified that $f_{2|1}(V_2|V_1)$ satisfies all the conditions of a PDF (i.e., it is non-negative, and satisfies the normalization condition, Eq. (3.96)). (A word on notation: '$|V_1$' is an abbreviation for '$|U_1 = V_1$,' and is read 'conditional on $U_1 = V_1$,' or 'given $U_1 = V_1$', or 'given V_1.')

For a function $Q(U_1, U_2)$, the *conditional mean* (conditional on V_1) $\langle Q|V_1\rangle$ is defined by

$$\langle Q(U_1, U_2)|U_1 = V_1\rangle \equiv \int_{-\infty}^{\infty} Q(V_1, V_2) f_{2|1}(V_2|V_1)\,\mathrm{d}V_2. \tag{3.97}$$

The concept of *independence* is of paramount importance. If U_1 and U_2 are independent, then knowledge of the value of either one of them provides no information about the other. Consequently, 'conditioning' has no effect, and the conditional and marginal PDFs are the same:

$$f_{2|1}(V_2|V_1) = f_2(V_2), \quad \text{for } U_1 \text{ and } U_2 \text{ independent.} \tag{3.98}$$

Hence (from Eq. (3.95)) the joint PDF is the product of the marginals:

$$f_{12}(V_1, V_2) = f_1(V_1) f_2(V_2), \quad \text{for } U_1 \text{ and } U_2 \text{ independent.} \tag{3.99}$$

Independent random variables are uncorrelated; but, in general, the converse is not true.

EXERCISES

3.14 Show that the properties of the joint PDF Eqs. (3.87)–(3.90) follow from the definitions of the CDF (Eq. (3.82)) and joint PDF (Eq. (3.86)).

3.15 Show that, for a function $R(U_1)$ of U_1 alone, the definition of the mean $\langle R(U_1)\rangle$ in terms of the joint PDF f_{12} (Eq. (3.91)) is consistent with its definition in terms of the marginal PDF f_1 (Eq. (3.20)).

3.16 By considering the quantity $(u_1/u_1' \pm u_2/u_2')^2$, establish the Cauchy–Schwarz inequality

$$-1 \le \rho_{12} \le 1, \tag{3.100}$$

where u_1' and u_2' are the standard deviations of U_1 and U_2, and ρ_{12} is the correlation coefficient.

3.17 Let U_1 and U_3 be uncorrelated random variables, and let U_2 be defined by

$$U_2 = a + bU_1 + cU_3, \tag{3.101}$$

where a, b, and c are constants. Show that the correlation coefficient ρ_{12} is

$$\rho_{12} = \frac{b}{(b^2 + c^2\langle u_3^2\rangle/\langle u_1^2\rangle)^{1/2}}. \tag{3.102}$$

Hence show that U_1 and U_2 are

(a) uncorrelated ($\rho_{12} = 0$) if b is zero and c is non-zero,
(b) perfectly correlated ($\rho_{12} = 1$) if c is zero and b is positive, and
(c) perfectly negatively correlated ($\rho_{12} = -1$) if c is zero and b is negative.

3.18 For the sum of two random variables, obtain the result

$$\mathrm{var}(U_1 + U_2) = \mathrm{var}(U_1) + \mathrm{var}(U_2) + 2\,\mathrm{cov}(U_1, U_2). \tag{3.103}$$

For the sum of N *independent* random variables obtain the result

$$\mathrm{var}\left(\sum_{i=1}^{N} U_i\right) = \sum_{i=1}^{N} \mathrm{var}(U_i). \tag{3.104}$$

3.19 Let U_1 be a standardized Gaussian random variable, and let U_2 be defined by $U_2 = |U_1|$. Sketch the possible values of (U_1, U_2) in the V_1–V_2 sample space. Show that U_1 and U_2 are uncorrelated. Argue that the conditional PDF of U_2 is

$$f_{2|1}(V_2|V_1) = \delta(V_2 - |V_1|), \tag{3.105}$$

and hence that U_2 and U_1 are not independent.

3.20 For any function $R(U_1)$, starting from Eq. (3.97), verify the result

$$\langle R(U_1)|V_1\rangle = R(V_1). \tag{3.106}$$

3.21 Show that the unconditional mean can be obtained from the conditional mean by

$$\langle Q(U_1, U_2)\rangle = \int_{-\infty}^{\infty} \langle Q|V_1\rangle f_1(V_1)\,\mathrm{d}V_1. \tag{3.107}$$

3.5 Normal and joint-normal distributions

In this section we introduce the *central-limit theorem* which (among other things) shows that the normal or Gaussian distribution (Eq. (3.41)) plays a central role in probability theory. Then the joint-normal distribution and its special properties are described. Many of the results given are most easily obtained via characteristic functions (Appendix I).

We begin by examining *ensemble averages*. Let U denote a component of velocity at a particular position and time in a repeatable turbulent-flow experiment, and let $U^{(n)}$ denote U on the nth repetition. Each repetition is performed under the same nominal conditions, and there is no dependence between different repetitions. Hence, the random variables $\{U^{(1)}, U^{(2)}, U^{(3)}, \ldots\}$ are independent and have the same distribution (i.e., that of U): they are said to be *independent and identically distributed* (i.i.d.).

The *ensemble average* (over N repetitions) is defined by

$$\langle U \rangle_N \equiv \frac{1}{N} \sum_{n=1}^{N} U^{(n)}. \tag{3.108}$$

The ensemble average is itself a random variable, and it is simple to show that its mean and variance are

$$\langle \langle U \rangle_N \rangle = \langle U \rangle, \tag{3.109}$$

$$\mathrm{var}(\langle U \rangle_N) = \frac{1}{N} \mathrm{var}(U) = \frac{\sigma_u^2}{N}. \tag{3.110}$$

Consequently (see Exercise 3.22) $\hat{U}$ defined by

$$\hat{U} = [\langle U \rangle_N - \langle U \rangle] N^{1/2} / \sigma_u \tag{3.111}$$

is a standardized random variable (i.e., $\langle \hat{U} \rangle = 0$, $\langle \hat{U}^2 \rangle = 1$).

The *central-limit theorem* states that, as N tends to infinity, the PDF of $\hat{U}$, $\hat{f}(V)$, tends to the standardized normal distribution

$$\hat{f}(V) = \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2} V^2), \tag{3.112}$$

(see Fig. 3.7 on page 47 and Exercise I.3 on page 709). This result depends on $\{U^{(1)}, U^{(2)}, \ldots, U^{(N)}\}$ being i.i.d. but the only restriction it places on the underlying random variable, U, is that it have finite variance.

We turn now to the *joint-normal distribution*, which is important both in probability theory and in turbulent flows. For example, in experiments on homogeneous turbulence the velocity components and a conserved passive scalar $\{U_1, U_2, U_3, \phi\}$ are found to be joint-normally distributed (see Fig. 5.46

on page 175). The definition and properties of the joint-normal distribution are now given for a general set of D random variables $\boldsymbol{U} = \{U_1, U_2, \ldots, U_D\}$. For $D = 2$ or 3, $\boldsymbol{U}$ can be thought of as components of velocity in a turbulent flow.

It is convenient to use matrix notation. The mean and fluctuation of the *random vector* $\boldsymbol{U}$ are denoted by

$$\boldsymbol{\mu} = \langle \boldsymbol{U} \rangle, \tag{3.113}$$

$$\boldsymbol{u} = \boldsymbol{U} - \langle \boldsymbol{U} \rangle. \tag{3.114}$$

The (symmetric $D \times D$) covariance matrix is then

$$\mathbf{C} = \langle \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \rangle, \tag{3.115}$$

If $\boldsymbol{U} = \{U_1, U_2, U_3\}$ is the velocity, then the covariance matrix is a second-order tensor with components $C_{ij} = \langle u_i u_j \rangle$.

If $\boldsymbol{U} = \{U_1, U_2, \ldots, U_D\}$ is joint-normally distributed, then (by definition) its joint PDF is

$$f(\boldsymbol{V}) = [(2\pi)^D \det(\mathbf{C})]^{-1/2} \exp\left[-\tfrac{1}{2}(\boldsymbol{V} - \boldsymbol{\mu})^{\mathrm{T}} \mathbf{C}^{-1} (\boldsymbol{V} - \boldsymbol{\mu})\right]. \tag{3.116}$$

Note that the $\boldsymbol{V}$-dependence of the joint PDF is contained in the quadratic form

$$g(\boldsymbol{V}) \equiv (\boldsymbol{V} - \boldsymbol{\mu})^{\mathrm{T}} \mathbf{C}^{-1} (\boldsymbol{V} - \boldsymbol{\mu}). \tag{3.117}$$

For $D = 2$, a constant value of g – corresponding to a constant probability density – is an ellipse in the V_1–V_2 plane. For $D = 3$, a constant-probability-surface is an ellipsoid in $\boldsymbol{V}$-space.

We now examine the pair $\{U_1, U_2\}$ of joint-normal random variables (i.e., $D = 2$) in more detail. Figure 3.18 shows a scatter plot and constant-probability-density lines for a particular choice of $\boldsymbol{\mu}$ and $\mathbf{C}$.

In terms of the variances $\langle u_1^2 \rangle$ and $\langle u_2^2 \rangle$, and the correlation coefficient ρ_{12}, the joint normal PDF (Eq. (3.116)) is

$$f_{12}(V_1, V_2) = \left[4\pi^2 \langle u_1^2 \rangle \langle u_2^2 \rangle (1 - \rho_{12}^2)\right]^{-1/2} \exp\left[\frac{-1}{2(1-\rho_{12}^2)} \times \left(\frac{(V_1 - \langle U_1 \rangle)^2}{\langle u_1^2 \rangle} - \frac{2\rho_{12}(V_1 - \langle U_1 \rangle)(V_2 - \langle U_2 \rangle)}{(\langle u_1^2 \rangle \langle u_2^2 \rangle)^{1/2}} + \frac{(V_2 - \langle U_2 \rangle)^2}{\langle u_2^2 \rangle} \right)\right]. \tag{3.118}$$

From this equation, the following properties can be deduced.

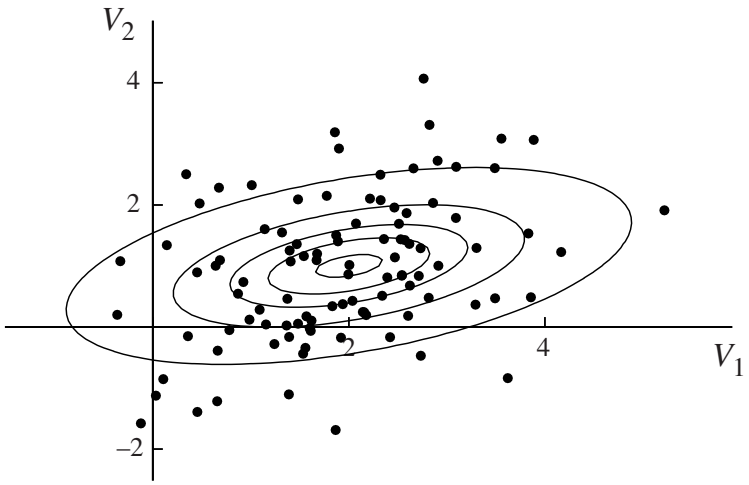

Fig. 3.18. A scatter plot and constant-probability density lines in the V_1–V_2 plane for joint-normal random variables $(U_1,\ U_2)$ with $\langle U_1\rangle = 2$, $\langle U_2\rangle = 1$, $\langle u_1^2\rangle = 1$, $\langle u_2^2\rangle = \frac{5}{16}$, and $\rho_{12} = 1/\sqrt{5}$.

(i) The marginal PDFs of U_1 and U_2 ($f_1(V_1)$ and $f_2(V_2)$) are Gaussian.

(ii) If U_1 and U_2 are uncorrelated (i.e., $\rho_{12} = 0$), then they are also independent (since then $f_{12}(V_1, V_2) = f_1(V_1)f_2(V_2)$). This is a special property of the joint-normal distribution: in general, lack of correlation does not imply independence.

(iii) The conditional mean of U_1 is

$$\langle U_1|U_2 = V_2\rangle = \langle U_1\rangle + \frac{\langle u_1 u_2\rangle}{\langle u_2^2\rangle}(V_2 - \langle U_2\rangle). \tag{3.119}$$

(iv) The conditional variance of U_1 is

$$\langle (U_1 - \langle U_1|V_2\rangle)^2|V_2\rangle = \langle u_1^2\rangle(1 - \rho_{12}^2). \tag{3.120}$$

(v) The conditional PDF $f_{1|2}(V_1|V_2)$ is Gaussian.

Returning to the general case of $\boldsymbol{U} = \{U_1, U_2, \ldots, U_D\}$ being joint normal, additional insight is gained by considering linear transformations of $\boldsymbol{U}$. An essential result (see Appendix I) is that, if $\boldsymbol{U}$ is joint normal, then a random vector $\hat{\boldsymbol{U}}$ formed by a general linear transformation of $\boldsymbol{U}$ is also joint normal.

Because the covariance matrix $\mathbf{C}$ is symmetric, it can be diagonalized by a unitary transformation, defined by a unitary matrix $\mathbf{A}$. (The properties of a unitary matrix are

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}, \tag{3.121}$$

where $\mathbf{I}$ is the $D \times D$ identity matrix.) That is, there is a unitary matrix $\mathbf{A}$

such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{C}\mathbf{A} = \mathbf{\Lambda}, \tag{3.122}$$

where $\mathbf{\Lambda}$ is the diagonal matrix containing the eigenvalues of $\mathbf{C}$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_D \end{bmatrix}. \tag{3.123}$$

Consequently the transformed random vector

$$\hat{\boldsymbol{u}} \equiv \mathbf{A}^{\mathrm{T}}\boldsymbol{u} \tag{3.124}$$

has a diagonal covariance matrix $\mathbf{\Lambda}$:

$$\hat{\mathbf{C}} = \langle \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^{\mathrm{T}} \rangle = \langle \mathbf{A}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\mathbf{A} \rangle = \mathbf{A}^{\mathrm{T}}\mathbf{C}\mathbf{A} = \mathbf{\Lambda}. \tag{3.125}$$

There are several observations to be made and results to be deduced from this transformation.

(i) If $\boldsymbol{U}$ is the velocity vector, then $\hat{\boldsymbol{u}}$ is the fluctuating velocity in a particular coordinate system – namely the principal axes of $\langle u_i u_j \rangle$.

(ii) The eigenvalues of $\mathbf{C}$, λ_i, are

$$\lambda_i = \langle \hat{u}_{(i)}\hat{u}_{(i)} \rangle \geq 0, \tag{3.126}$$

(where bracketed suffixes are excluded from the summation convention). Thus, since each eigenvalue is non-negative, $\mathbf{C}$ is symmetric positive semi-definite.

(iii) That the covariance matrix $\hat{\mathbf{C}}$ is diagonal indicates that the transformed random variables $\{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_D\}$ are uncorrelated.

These three observations apply irrespective of whether $\boldsymbol{U}$ is joint normal. In addition we have the following

(iv) If $\boldsymbol{U}$ is joint normal, then $\{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_D\}$ are *independent* Gaussian random variables.

EXERCISES

3.22 From the definition of the ensemble average (Eq. (3.108)) show that

$$\langle \langle U \rangle_N^2 \rangle = \langle U \rangle^2 + \frac{1}{N}\,\mathrm{var}(U), \tag{3.127}$$

and hence verify Eq. (3.110). Hint:

$$\langle U\rangle_N^2 = \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} U^{(n)}U^{(m)}. \tag{3.128}$$

3.23 Obtain an explicit expression for the kurtosis of $\langle U\rangle_N$ in terms of N and the kurtosis of U. Comment on the result in light of the central-limit theorem.

3.24 Show that, for large N, the ensemble mean (Eq. (3.108)) can be written

$$\langle U\rangle_N = \langle U\rangle + N^{-1/2}u'\xi,$$

where $u' = \text{sdev}(U)$ and ξ is a standardized Gaussian random variable.

3.25 Let $\boldsymbol{U}$ be a joint-normal random vector with mean $\boldsymbol{\mu}$ and positive-definite covariance matrix $\mathbf{C} = \mathbf{A}\boldsymbol{\Lambda}\mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}$ is unitary and $\boldsymbol{\Lambda}$ is diagonal. Show that the random variable

$$\hat{\boldsymbol{u}} \equiv \mathbf{C}^{-1/2}(\boldsymbol{U} - \boldsymbol{\mu})$$

is a *standardized joint normal*, i.e., it has mean zero, identity covariance, and joint PDF

$$\hat{f}(\hat{\boldsymbol{V}}) = \left(\frac{1}{2\pi}\right)^{D/2} \exp\left(-\tfrac{1}{2}\,\hat{\boldsymbol{V}}^{\mathrm{T}}\hat{\boldsymbol{V}}\right). \tag{3.129}$$

3.26 A Gaussian random-number generator produces a sequence of independent standardized Gaussian random numbers: $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$, $\boldsymbol{\xi}^{(3)}$, ... How can these be used to generate a joint-normal random vector $\boldsymbol{U}$ with specified mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$?

(Hint: this can be achieved in a number of ways, the best of which involves the *Cholesky decomposition*, i.e., a symmetric semi-definite matrix can be decomposed as $\mathbf{C} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$, where $\mathbf{L}$ is lower triangular.)

3.6 Random processes

As an example of a random variable, we considered (in Section 3.2) a component of velocity U in a repeatable turbulent-flow experiment, at a particular location and time (relative to the initiation of the experiment). The random variable U is completely characterized by its PDF, $f(V)$. Consider now the same velocity, but as a function of time, i.e., $U(t)$. Such a time-dependent random variable is called a *random process*. Figure 3.19 illustrates

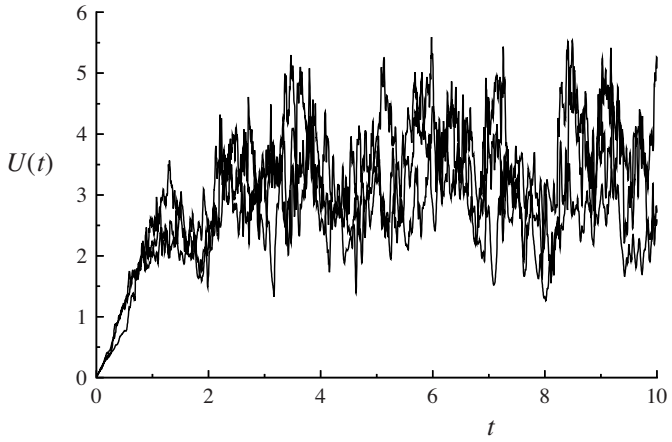

Fig. 3.19. Sample paths of $U(t)$ from three repetitions of a turbulent-flow experiment.

sample paths (i.e., values of $U(t)$) obtained in different repetitions of the experiment.

How can a random process be characterized? At each point in time, the random variable $U(t)$ is characterized by its *one-time CDF*

$$F(V,t) \equiv P\{U(t) < V\}, \tag{3.130}$$

or, equivalently, by the one-time PDF

$$f(V;t) \equiv \frac{\partial F(V,t)}{\partial V}. \tag{3.131}$$

However, these quantities contain no joint information about $U(t)$ at two or more times. To illustrate this limitation, Fig. 3.20 shows sample paths of five different random processes, each with the same one-time PDF. Clearly, radically different behavior (qualitatively and quantitatively) is possible, but is not represented by the one-time PDF. The N-time joint CDF of the process $U(t)$ is defined by

$$F_N(V_1,t_1;V_2,t_2;\ldots;V_N,t_N) \equiv P\{U(t_1) < V_1, U(t_2) < V_2, \ldots, U(t_N) < V_N\}, \tag{3.132}$$

where $\{t_1,t_2,\ldots,t_N\}$ are specified time points, and $f_N(V_1,t_1;V_2,t_2;\ \ldots;V_N,t_N)$ is the corresponding N-time joint PDF. To completely characterize the random process, it is necessary to know this joint PDF for *all* instants of time, which is, in general, an impossible task.

Considerable simplification occurs if the process is *statistically stationary*, as are many (but certainly not all) turbulent flows. A process is statistically stationary if all multi-time statistics are invariant under a shift in time, i.e.,

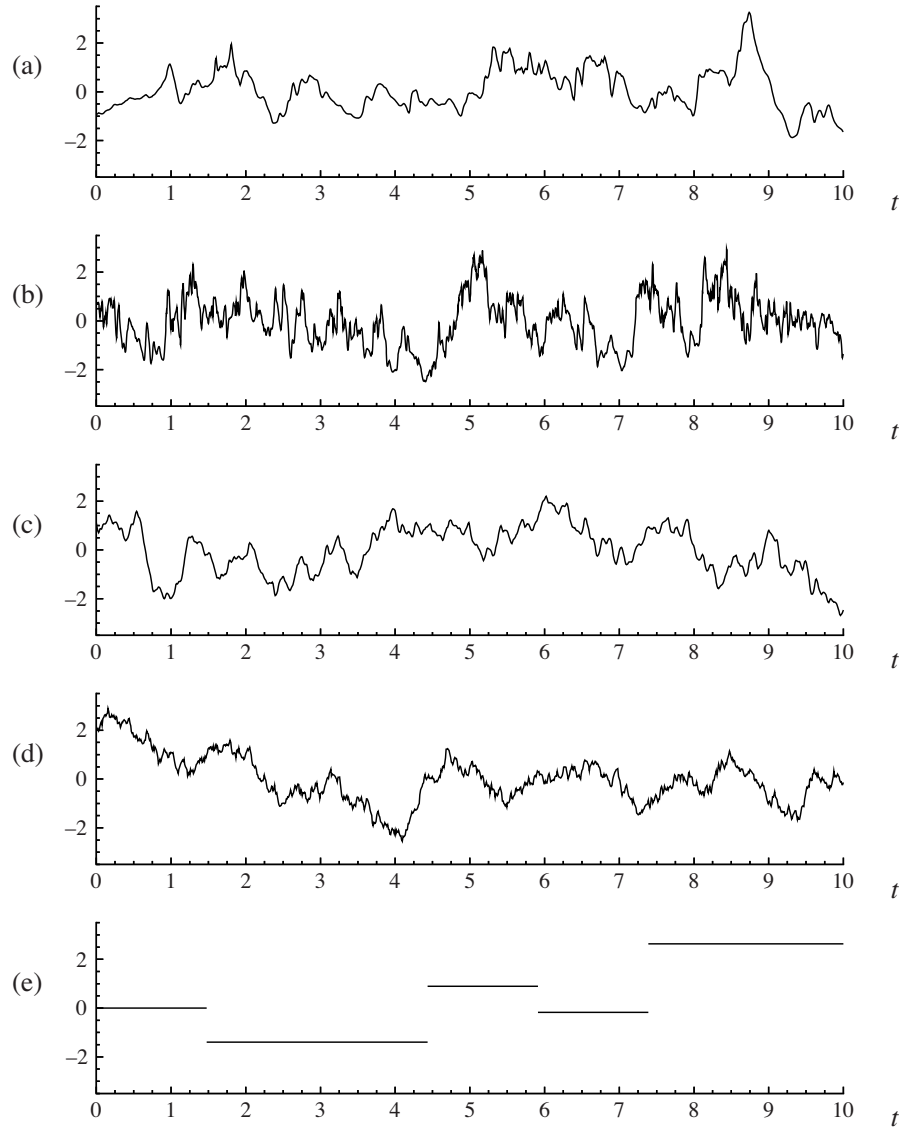

Fig. 3.20. Sample paths of five statistically stationary random processes. The one-time PDF of each is a standardized Gaussian. (a) A measured turbulent velocity. (b) A measured turbulent velocity of a higher frequency than that of (a). (c) A Gaussian process with the same spectrum as that of (a). (d) An Ornstein–Uhlenbeck process (see Chapter 12) with the same integral timescale as that of (a). (e) A jump process with the same spectrum as that of (d).

for all positive time intervals T, and all choices of $\{t_1, t_2, \ldots, t_N\}$, we have

$$f(V_1, t_1 + T; V_2, t_2 + T; \ldots; V_N, t_N + T) = f(V_1, t_1; V_2, t_2, \ldots, V_N, t_N). \tag{3.133}$$

After a laminar flow has been initiated, it can pass through an initial transient period and then reach a steady state, in which the flow variables

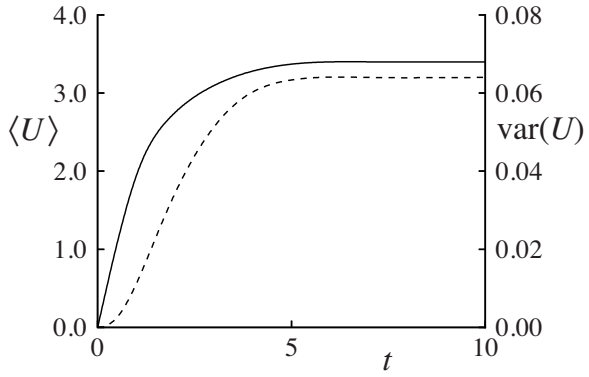

Fig. 3.21. The mean $\langle U(t)\rangle$ (solid line) and variance $\langle u(t)^2\rangle$ of the process shown in Fig. 3.19.

are independent of time. A turbulent flow, after an initial transient period, can reach a statistically stationary state in which, even though the flow variables (e.g., $U(t)$) vary with time, the statistics are independent of time. This is the case for the process shown in Fig. 3.19. The mean $\langle U(t)\rangle$ and variance $\langle u(t)^2\rangle$ of this process are shown in Fig. 3.21. Evidently, after $t \approx 5$, the statistics become independent of time, even though the process itself $U(t)$ continues to vary significantly.

For a statistically stationary process, the simplest multi-time statistic that can be considered is the *autocovariance*

$$R(s) \equiv \langle u(t)u(t+s)\rangle, \tag{3.134}$$

or, in normalized form, the *autocorrelation function*

$$\rho(s) \equiv \langle u(t)u(t+s)\rangle / \langle u(t)^2\rangle, \tag{3.135}$$

where $u(t) \equiv U(t) - \langle U\rangle$ is the fluctuation. (Note that, in view of the assumed statistical stationarity, the mean $\langle U\rangle$, the variance $\langle u^2\rangle$, $R(s)$, and $\rho(s)$ do not depend upon t.) The autocorrelation function is the correlation coefficient between the process at times t and $t+s$. Consequently it has the properties

$$\rho(0) = 1, \tag{3.136}$$

$$|\rho(s)| \leq 1. \tag{3.137}$$

Further, putting $t' = t + s$, we obtain

$$\begin{aligned} \rho(s) &= \langle u(t'-s)u(t')\rangle / \langle u^2\rangle \\ &= \rho(-s), \end{aligned} \tag{3.138}$$

i.e., $\rho(s)$ is an even function.

If $U(t)$ is periodic with period T (i.e., $U(t+T) = U(t)$), then so also is

$\rho(s)$ (i.e., $\rho(s+T)=\rho(s)$). However, for processes arising in turbulent flows, we expect the correlation to diminish as the lag time s increases. Usually $\rho(s)$ decreases sufficiently rapidly that the integral

$$\bar{\tau} \equiv \int_0^\infty \rho(s)\,\mathrm{d}s \tag{3.139}$$

converges: then $\bar{\tau}$ is the *integral timescale* of the process.

Figure 3.22 shows the autocorrelation functions for the five processes given in Fig. 3.20. Notice in particular that the high-frequency process (b) has a narrower autocorrelation function (and hence a smaller $\bar{\tau}$) than does the low-frequency process (a). By construction, process (c) has the same autocorrelation as that of (a). Processes (d) and (e) both have the autocorrelation function $\rho(s)=\exp(-|s|/\bar{\tau})$, with the same integral timescale as that of process (a). Hence, apart from (b), all the processes have the same integral timescale.

The autocovariance $R(s) \equiv \langle u(t)u(t+s)\rangle = \langle u(t)^2\rangle\rho(s)$ and (twice) the *frequency spectrum* $E(\omega)$ form a Fourier-transform pair:

$$\begin{aligned} E(\omega) &\equiv \frac{1}{\pi}\int_{-\infty}^{\infty} R(s)e^{-i\omega s}\,\mathrm{d}s \\ &= \frac{2}{\pi}\int_0^\infty R(s)\cos(\omega s)\,\mathrm{d}s, \end{aligned} \tag{3.140}$$

and

$$\begin{aligned} R(s) &= \frac{1}{2}\int_{-\infty}^{\infty} E(\omega)e^{i\omega s}\,\mathrm{d}\omega \\ &= \int_0^\infty E(\omega)\cos(\omega s)\,\mathrm{d}\omega. \end{aligned} \tag{3.141}$$

(The definitions and properties of Fourier transforms are given in Appendix D.) Clearly $R(s)$ and $E(\omega)$ contain the same information, just in different forms. Because $R(s)$ is real and even, so also is $E(\omega)$.

As discussed more fully in Appendix E, the velocity fluctuation $u(t)$ has a spectral representation as the weighted sum of *Fourier modes* of different frequencies ω, i.e., $e^{i\omega t}=\cos(\omega t)+i\sin(\omega t)$. The fundamental property of the frequency spectrum is that (for $\omega_\mathrm{a}<\omega_\mathrm{b}$) the integral

$$\int_{\omega_\mathrm{a}}^{\omega_\mathrm{b}} E(\omega)\,\mathrm{d}\omega \tag{3.142}$$

is the contribution to the variance $\langle u(t)^2\rangle$ of all modes in the frequency range

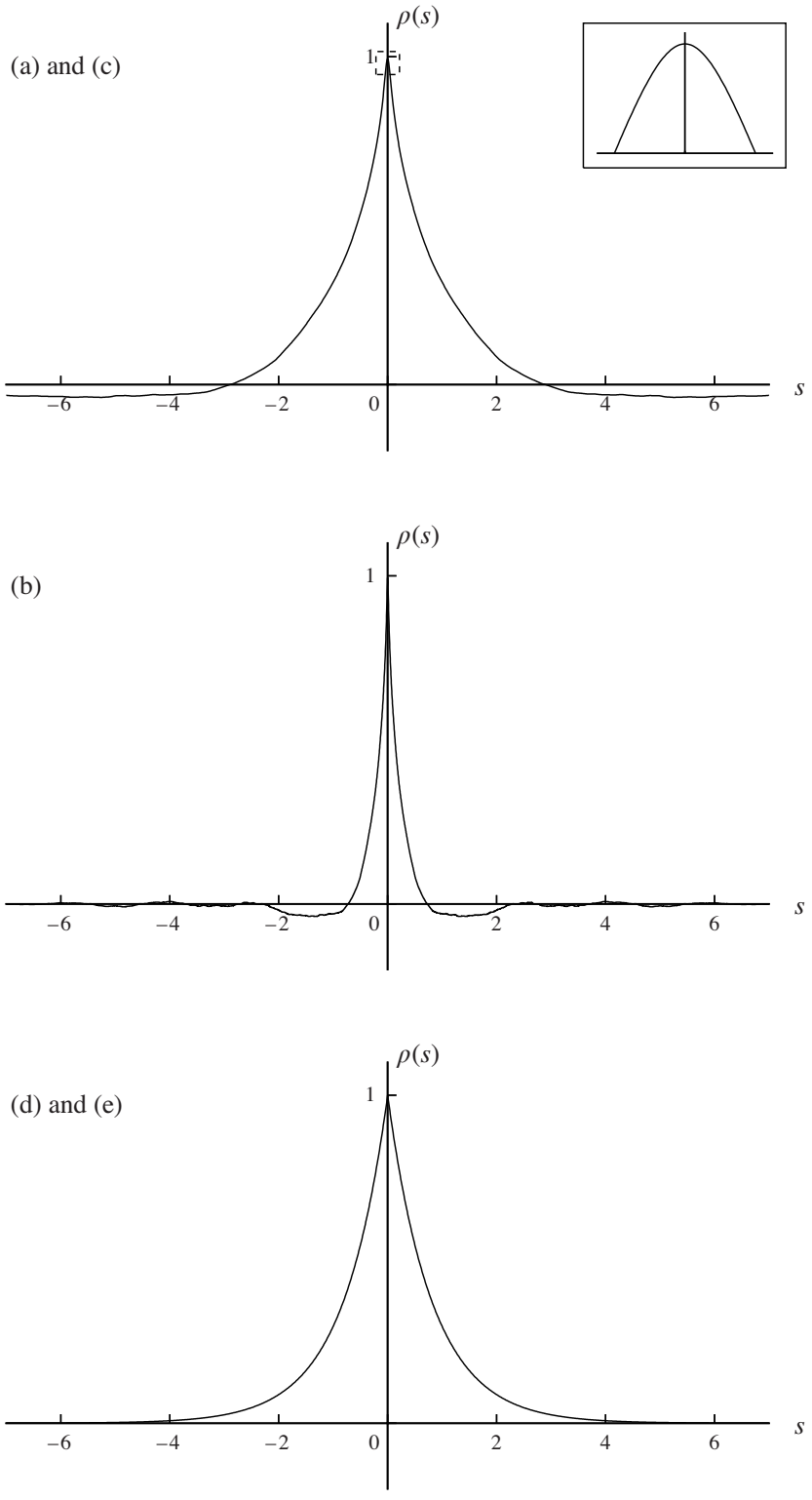

Fig. 3.22. Autocorrelation functions of the processes shown in Fig. 3.20. As the inset shows, for processes (a) and (c) the autocorrelation function is smooth at the origin.

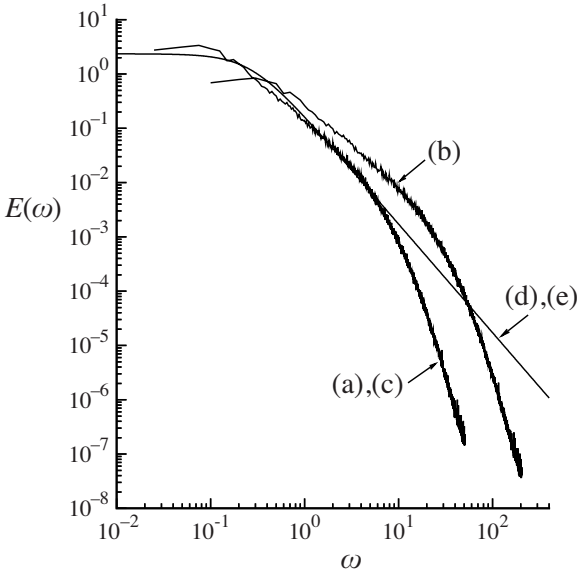

Fig. 3.23. Spectra of processes shown in Fig. 3.20.

$\omega_a \leq \omega < \omega_b$. In particular the variance is

$$R(0) = \langle u(t)^2 \rangle = \int_0^\infty E(\omega)\,\mathrm{d}\omega, \tag{3.143}$$

as is evident from Eq. (3.141) with $s = 0$.

A further simple connection between the spectrum and the autocorrelation is that the integral timescale is given by

$$\bar{\tau} = \frac{\pi E(0)}{2\langle u^2 \rangle}, \tag{3.144}$$

as is readily verified by setting $\omega = 0$ in Eq. (3.140). A more complete explanation of the spectral representation and interpretation of the frequency spectrum is given in Appendix E.

Figure 3.23 shows the spectra of the stationary random processes given in Fig. 3.20. The high-frequency process (b), having a smaller integral timescale than that of process (a), has a correspondingly smaller value of the spectrum at the origin (Eq. 3.144) – but its spectrum extends to higher frequencies.

In practice, the autocorrelation function or the spectrum is usually the only quantity used to characterize the multi-time properties of a random process. However, it should be appreciated that the one-time PDF and the autocorrelation function provide only a partial characterization of the process. This point is amply demonstrated by processes (d) and (e) in Fig. 3.20. The two processes are qualitatively quite different and yet they have the same one-time PDF (Gaussian) and the same autocorrelation

function ($\rho(s) = e^{-|s|/\bar{\tau}}$). To repeat, in general, *the one-time PDF and the autocorrelation function do not completely characterize a random process.*

A *Gaussian process* is an important but very special case. If a process is Gaussian then, by definition, the general N-time PDF (Eq. (3.133)) is joint normal. Now the joint-normal distribution is fully characterized by its means $\langle U(t_n)\rangle$, and its covariances $\langle u(t_n)u(t_m)\rangle$. For a statistically stationary process, we have

$$\langle u(t_n)u(t_m)\rangle = R(t_n - t_m) = \langle u(t)^2\rangle\ \rho(t_n - t_m). \tag{3.145}$$

Hence a statistically stationary Gaussian process is completely characterized by its mean $\langle U(t)\rangle$, its variance $\langle u(t)^2\rangle$, and the autocorrelation function $\rho(s)$ (or equivalently the spectrum $E(\omega)$).

In Fig. 3.20, process (c) is defined to be the Gaussian process with the same spectrum as that of the turbulent velocity, process (a). Some differences between processes (a) and (c) may be discernible; and these differences can be clearly revealed by, for example, examining the sample paths of $\ddot{U}(t) \equiv \mathrm{d}^2U(t)/\mathrm{d}t^2$, see Fig. 3.24. For the Gaussian process (c) it follows that $\ddot{U}(t)$ is also Gaussian and so the kurtosis of $\ddot{U}(t)$ is 3. However, for the turbulent velocity, process (a), $\ddot{U}(t)$ is far from Gaussian, and has a kurtosis of 11.

Random processes arising from turbulence (e.g., process (a)) are differentiable, i.e., for each sample path the following limit exists:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \lim_{\Delta t \downarrow 0}\left(\frac{U(t+\Delta t) - U(t)}{\Delta t}\right). \tag{3.146}$$

In this case, taking the mean and taking the limit commute, so that

$$\begin{aligned}\left\langle \frac{\mathrm{d}U(t)}{\mathrm{d}t}\right\rangle &= \left\langle \lim_{\Delta t \downarrow 0}\left(\frac{U(t+\Delta t) - U(t)}{\Delta t}\right)\right\rangle \\ &= \lim_{\Delta t \downarrow 0}\left(\frac{\langle U(t+\Delta t)\rangle - \langle U(t)\rangle}{\Delta t}\right) \\ &= \frac{\mathrm{d}\langle U(t)\rangle}{\mathrm{d}t}.\end{aligned} \tag{3.147}$$

Synthetic processes (such as processes (d) and (e)) need not be differentiable (i.e., the limit Eq. (3.146) does not exist). It may be observed that the spectra of these processes decay as $E(\omega) \sim \omega^{-2}$ at high frequencies, and that (correspondingly) their autocorrelation function $\rho(s) = e^{-|s|/\bar{\tau}}$ is not differentiable at the origin.

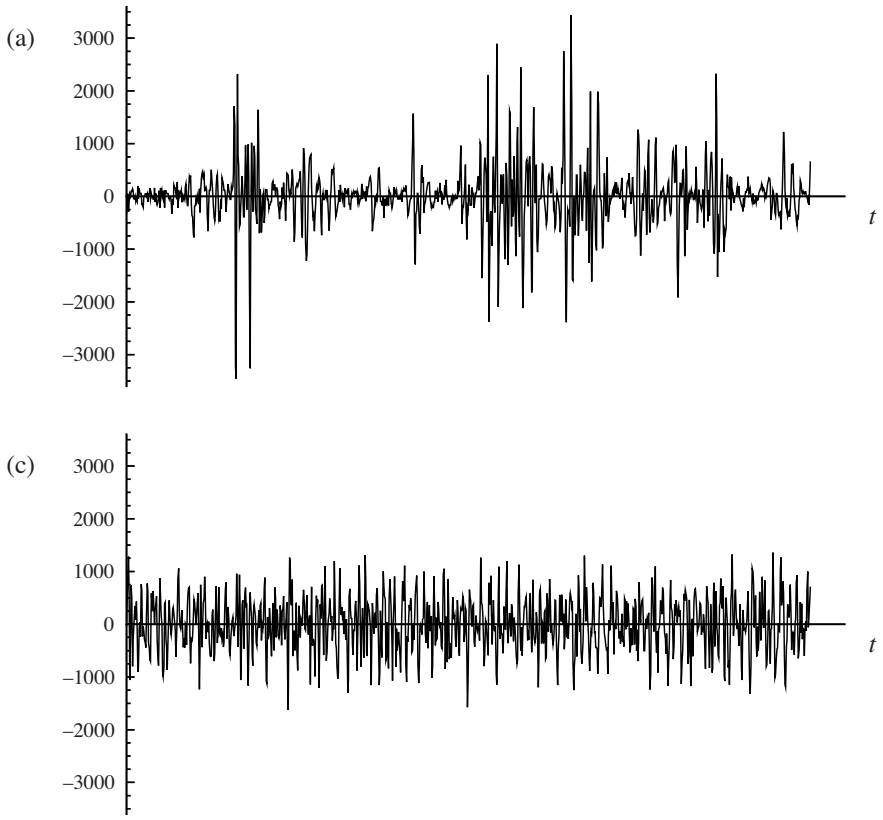

Fig. 3.24. Sample paths of $\ddot{U}(t)$ for processes (a) and (c) shown in Fig. 3.20.

Process (d) is an *Ornstein–Uhlenbeck process*, which is the canonical example of a *diffusion process*. Such processes are used in PDF methods and are described in Chapter 12 and Appendix J.

EXERCISES

In the following exercises, $u(t)$ is a zero-mean, statistically stationary, differentiable random process with autocovariance $R(s)$, autocorrelation function $\rho(s)$, and spectrum $E(\omega)$.

3.27 Show that $u(t)$ and $\dot{u}(t)$ are uncorrelated, and that $u(t)$ and $\ddot{u}(t)$ are negatively correlated.

3.28 Show that

$$\left\langle u^2 \frac{\mathrm{d}^3 u}{\mathrm{d}t^3} \right\rangle = -2\langle u\dot{u}\ddot{u}\rangle = 2\langle(\dot{u})^3\rangle + 2\langle u\dot{u}\ddot{u}\rangle = \langle(\dot{u})^3\rangle. \tag{3.148}$$

3.29 Show that at the origin ($s = 0$) $\mathrm{d}R(s)/\mathrm{d}s$ is zero, and $\mathrm{d}^2R(s)/\mathrm{d}s^2$ is negative.

3.30 Show that the autocovariance $B(s)$ of the process $\dot{u}(t)$ is

$$B(s) = -\frac{\mathrm{d}^2 R(s)}{\mathrm{d}s^2}. \tag{3.149}$$

3.31 Show that the integral timescale of $\dot{u}(t)$ is zero.

3.32 Show that the spectrum of $\dot{u}(t)$ is $\omega^2 E(\omega)$.

3.33 If $u(t)$ is a Gaussian process, show that

$$\langle \dot{u}(t) | u(t) = v \rangle = 0, \tag{3.150}$$

$$\langle \ddot{u}(t) | u(t) = v \rangle = -v \langle \dot{u}(t)^2 \rangle / \langle u^2 \rangle. \tag{3.151}$$

3.7 Random fields

In a turbulent flow, the velocity $\boldsymbol{U}(\boldsymbol{x}, t)$ is a time-dependent random vector field. It can be described – i.e., partially characterized – by extensions of the tools presented in the previous sections.

One-point statistics

The one-point, one-time joint CDF of velocity is

$$F(\boldsymbol{V}, \boldsymbol{x}, t) = P\{U_i(\boldsymbol{x}, t) < V_i,\ i = 1, 2, 3\}, \tag{3.152}$$

and then the joint PDF is

$$f(\boldsymbol{V}; \boldsymbol{x}, t) = \frac{\partial^3 F(\boldsymbol{V}, \boldsymbol{x}, t)}{\partial V_1\, \partial V_2\, \partial V_3}. \tag{3.153}$$

At each point and time this PDF fully characterizes the random velocity vector, but it contains no joint information at two or more times or positions. In terms of this PDF, the mean velocity field is

$$\langle \boldsymbol{U}(\boldsymbol{x}, t) \rangle = \int_{-\infty}^{\infty}\!\!\int\!\!\int \boldsymbol{V} f(\boldsymbol{V}; \boldsymbol{x}, t)\, \mathrm{d}V_1\, \mathrm{d}V_2\, \mathrm{d}V_3 \tag{3.154}$$

$$= \int \boldsymbol{V} f(\boldsymbol{V}; \boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{V}. \tag{3.155}$$

The second line of this equation introduces an abbreviated notation: $\int (\,)\, \mathrm{d}\boldsymbol{V}$ is written for

$$\int_{-\infty}^{\infty}\!\!\int\!\!\int (\,)\, \mathrm{d}V_1\, \mathrm{d}V_2\, \mathrm{d}V_3.$$

The fluctuating velocity field is defined by

$$\boldsymbol{u}(\boldsymbol{x},t) \equiv \boldsymbol{U}(\boldsymbol{x},t) - \langle \boldsymbol{U}(\boldsymbol{x},t)\rangle. \tag{3.156}$$

The (one-point, one-time) covariance of the velocity is $\langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t)\rangle$. For reasons given in the next chapter, these covariances are called *Reynolds stresses*, and are written $\langle u_i u_j\rangle$, with the dependences on $\boldsymbol{x}$ and t being understood.

A word on notation: the semi-colon in $f(\boldsymbol{V};\boldsymbol{x},t)$ indicates that f is a *density* with respect to the sample-space variables that appear to the left of the semi-colon (i.e., V_1, V_2, and V_3), whereas f is a *function* with respect to the remaining variables (i.e., x_1, x_2, x_3, and t). This distinction is useful because densities and functions have different transformation properties (see Exercise 3.9 on page 49).

Turbulent velocity fields are differentiable, and (as discussed in Section 3.6) differentiation and taking the mean commute:

$$\left\langle \frac{\partial U_i}{\partial t} \right\rangle = \frac{\partial \langle U_i\rangle}{\partial t}, \tag{3.157}$$

$$\left\langle \frac{\partial U_i}{\partial x_j} \right\rangle = \frac{\partial \langle U_i\rangle}{\partial x_j}. \tag{3.158}$$

N-point statistics

The N-point, N-time joint PDF can be defined as a simple extension of Eq. (3.132). Let $\{(\boldsymbol{x}^{(n)},t^{(n)}), n = 1,2,\ldots,N\}$ be a specified set of positions and times. Then we define

$$f_N(\boldsymbol{V}^{(1)},\boldsymbol{x}^{(1)},t^{(1)},\boldsymbol{V}^{(2)},\boldsymbol{x}^{(2)},t^{(2)};\ldots;\boldsymbol{V}^{(N)},\boldsymbol{x}^{(N)},t^{(N)}) \tag{3.159}$$

to be the joint PDF of $\boldsymbol{U}(\boldsymbol{x},t)$ at these N space–time points. To determine this N-point PDF for *all* space–time points is obviously impossible, and hence in practice a random velocity field cannot be fully characterized.

Turbulent velocity fields are found *not* to be Gaussian: a Gaussian field is fully characterized by the mean $\langle \boldsymbol{U}(\boldsymbol{x},t)\rangle$ and the autocovariance $\langle u_i(\boldsymbol{x}^{(1)},t^{(1)})u_j(\boldsymbol{x}^{(2)},t^{(2)})\rangle$.

Statistical stationarity and homogeneity

The random field $\boldsymbol{U}(\boldsymbol{x},t)$ is *statistically stationary* if all statistics are invariant under a shift in time. In terms of the N-point PDF, this means that f_N is unchanged if $(\boldsymbol{x}^{(n)},t^{(n)})$ is replaced by $(\boldsymbol{x}^{(n)},t^{(n)}+T)$ for all N points, where T is the time shift.

Similarly, the field is *statistically homogeneous* if all statistics are invariants

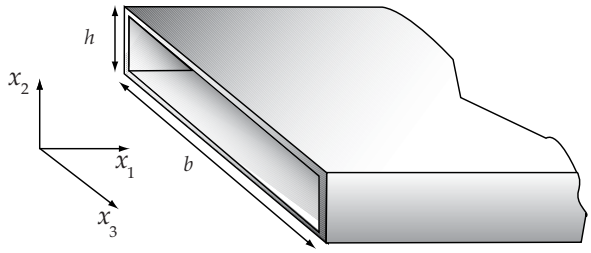

Fig. 3.25. A sketch of a turbulent-channel-flow apparatus.

under a shift in position. Then f_N is unchanged if $(\boldsymbol{x}^{(n)}, t^{(n)})$ is replaced by $(\boldsymbol{x}^{(n)} + \boldsymbol{X}, t^{(n)})$, for all N points, where $\boldsymbol{X}$ is the shift in position. If the velocity field $\boldsymbol{U}(\boldsymbol{x}, t)$ is statistically homogeneous, it follows that the mean velocity $\langle \boldsymbol{U} \rangle$ is uniform; and, with an appropriate choice of frame, $\langle \boldsymbol{U} \rangle$ can be taken to be zero. The definition of *homogeneous turbulence* is less restrictive: specifically, in homogeneous turbulence the *fluctuating* velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is statistically homogeneous. It is consistent with this definition for the mean velocity gradients $\partial \langle U_i \rangle / \partial x_j$ to be non-zero, but uniform (see Section 5.4.5). A good approximation to homogeneous turbulence can be achieved in wind-tunnel experiments; and homogeneous turbulence is the simplest class of flows to study using direct numerical simulation.

In a similar way, turbulent flows can be statistically two-dimensional or one-dimensional. For example, Fig. 3.25 is a sketch of a channel flow apparatus. For a large aspect ratio ($b/h \gg 1$), and remote from the end walls ($|x_3|/b \ll 1$), the statistics of the flow vary little in the spanwise (x_3) direction. To within an approximation, then, the velocity field $\boldsymbol{U}(\boldsymbol{x}, t)$ is statistically two-dimensional – statistics being independent of x_3. Sufficiently far down the channel ($x_1/h \gg 1$) the flow becomes (statistically) fully developed. Then the velocity field is statistically one-dimensional, with statistics being independent both of x_1 and of x_3. Similarly, the turbulent flow in a pipe is *statistically axisymmetric* in that (in polar-cylindrical coordinates) all statistics are independent of the circumferential coordinate.

It should be emphasized that, even if a flow is statistically homogeneous or one-dimensional, nevertheless all three components of $\boldsymbol{U}(\boldsymbol{x}, t)$ vary in all three coordinate directions and time. It is only the statistics that are independent of some coordinate directions.

Isotropic turbulence

A statistically homogeneous field $\boldsymbol{U}(\boldsymbol{x}, t)$ is, by definition, statistically invariant under translations (i.e., shifts in the origin of the coordinate system). If the field is also statistically invariant under rotations and reflections of the

coordinate system, then it is (statistically) *isotropic*. The concept of isotropy is extremely important in turbulence: hundreds of wind-tunnel experiments have been performed on (approximately) isotropic turbulence, and much of turbulence theory centers on it. In terms of the N-point PDF (Eq. (3.159)), in isotropic turbulence f_N is unchanged if $\boldsymbol{U}(\boldsymbol{x}^{(n)},\ t^{(n)})$ is replaced by $\overline{\boldsymbol{U}}(\overline{\boldsymbol{x}}^{(n)}, t^{(n)})$, where $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{U}}$ denote the position and velocity in any coordinate system obtained by rotation and reflections of the coordinate axes.

Two-point correlation

The simplest statistic containing some information on the spatial structure of the random field is the two-point, one-time autocovariance

$$R_{ij}(\boldsymbol{r}, \boldsymbol{x}, t) \equiv \langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x} + \boldsymbol{r}, t) \rangle, \tag{3.160}$$

which is often referred to as the *two-point correlation*. From this it is possible to define various integral lengthscales, for example

$$L_{11}(\boldsymbol{x}, t) \equiv \frac{1}{R_{11}(0, \boldsymbol{x}, t)} \int_0^\infty R_{11}(\boldsymbol{e}_1 r, \boldsymbol{x}, t)\, \mathrm{d}r, \tag{3.161}$$

where $\boldsymbol{e}_1$ is the unit vector in the x_1-coordinate direction.

Wavenumber spectra

For homogeneous turbulence the two-point correlation $R_{ij}(\boldsymbol{r}, t)$ is independent of $\boldsymbol{x}$, and the information it contains can be re-expressed in terms of the *wavenumber spectrum*. The spatial Fourier mode

$$e^{i\boldsymbol{\kappa}\cdot\boldsymbol{x}} = \cos(\boldsymbol{\kappa}\cdot\boldsymbol{x}) + i \sin(\boldsymbol{\kappa}\cdot\boldsymbol{x}), \tag{3.162}$$

is a function that varies sinusoidally (with wavelength $\ell = 2\pi/|\boldsymbol{\kappa}|$) in the direction of the wavenumber vector $\boldsymbol{\kappa}$, and that is constant in planes normal to $\boldsymbol{\kappa}$. The velocity spectrum tensor $\Phi_{ij}(\boldsymbol{\kappa}, t)$ is the Fourier transform of the two-point correlation

$$\Phi_{ij}(\boldsymbol{\kappa}, t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{r}} R_{ij}(\boldsymbol{r}, t)\, \mathrm{d}\boldsymbol{r}, \tag{3.163}$$

and the inverse transform is

$$R_{ij}(\boldsymbol{r}, t) = \iiint_{-\infty}^{\infty} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}} \Phi_{ij}(\boldsymbol{\kappa}, t)\, \mathrm{d}\boldsymbol{\kappa}, \tag{3.164}$$

where $\mathrm{d}\boldsymbol{r}$ and $\mathrm{d}\boldsymbol{\kappa}$ are written for $\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3$ and $\mathrm{d}\kappa_1\,\mathrm{d}\kappa_2\,\mathrm{d}\kappa_3$, respectively. Setting $\boldsymbol{r} = 0$ in this equation yields

$$R_{ij}(0,t) = \langle u_i u_j \rangle = \iiint_{-\infty}^{\infty} \Phi_{ij}(\boldsymbol{\kappa}, t)\,\mathrm{d}\boldsymbol{\kappa}. \tag{3.165}$$

and so $\Phi_{ij}(\boldsymbol{\kappa}, t)$ represents the contribution to the covariance $\langle u_i u_j \rangle$ of velocity modes with wavenumber $\boldsymbol{\kappa}$.

The two-point correlation and the spectrum contain two different kinds of directional information. The dependences of $R_{ij}(\boldsymbol{r}, t)$ on $\boldsymbol{r}$, and of $\Phi_{ij}(\boldsymbol{\kappa}, t)$ on $\boldsymbol{\kappa}$, give information about the directional dependence of correlation; while the components of R_{ij} and Φ_{ij} give information about the directions of the velocities.

A useful quantity, especially for qualitative discussions, is the *energy spectrum function*:

$$E(\kappa, t) \equiv \iiint_{-\infty}^{\infty} \tfrac{1}{2}\Phi_{ii}(\boldsymbol{\kappa}, t)\delta(|\boldsymbol{\kappa}| - \kappa)\,\mathrm{d}\boldsymbol{\kappa}, \tag{3.166}$$

which may be viewed as $\Phi_{ij}(\boldsymbol{\kappa}, t)$ stripped of all directional information. Integration of Eq. (3.166) over all scalar wavenumbers, κ, yields

$$\int_0^{\infty} E(\kappa, t)\,\mathrm{d}\kappa = \tfrac{1}{2}R_{ii}(0,t) = \tfrac{1}{2}\langle u_i u_i \rangle. \tag{3.167}$$

Thus, $E(\kappa, t)\,\mathrm{d}\kappa$ represents the contribution to the turbulent kinetic energy $\frac{1}{2}\langle u_i u_i \rangle$ from all modes with $|\boldsymbol{\kappa}|$ in the range $\kappa \le |\boldsymbol{\kappa}| < \kappa + \mathrm{d}\kappa$. Velocity spectra in turbulence are examined in some detail in Section 6.5.

EXERCISES

3.34 From the substitution $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$ and the definition of the two-point correlation (Eq. (3.160)), show that

$$R_{ij}(\boldsymbol{r}, \boldsymbol{x}, t) = R_{ji}(-\boldsymbol{r}, \boldsymbol{x}', t), \tag{3.168}$$

and hence, for a statistically homogeneous field,

$$R_{ij}(\boldsymbol{r}, t) = R_{ji}(-\boldsymbol{r}, t). \tag{3.169}$$

3.35 If $\boldsymbol{u}(\boldsymbol{x}, t)$ is divergence-free (i.e., $\nabla \cdot \boldsymbol{u} = 0$), show that the two-point

correlation (Eq. (3.160)) satisfies

$$\frac{\partial}{\partial r_j} R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = 0. \tag{3.170}$$

Show that, if, in addition, $\boldsymbol{u}(\boldsymbol{x}, t)$ is statistically homogeneous, then

$$\frac{\partial}{\partial r_j} R_{ij}(\boldsymbol{r}, t) = \frac{\partial}{\partial r_i} R_{ij}(\boldsymbol{r}, t) = 0. \tag{3.171}$$

3.8 Probability and averaging

Having developed the tools to describe random variables, random processes, and random fields, we now return to the starting point in order to clarify the notion of *probability*, on which everything has been built. Physical quantities such as density and velocity are defined operationally (e.g., in Section 2.1), so that (at least in principle) their values can be determined by measurement. Operational definitions of probability – for example, in terms of time averages or ensemble averages – although they are often used, are unsatisfactory. Instead, in modern treatments, probability theory is *axiomatic*. The purpose of this section is to describe this axiomatic approach, and to explain the connection to measurable quantities (such as time averages). For the sake of simplicity, we start the discussion in the context of a coin-tossing experiment.

Consider a coin that can be tossed any number of times, with the two possible outcomes 'heads' and 'tails.' We *define* the variable p to be the probability of 'heads.' (It is assumed that each toss is statistically independent and indistinguishable from every other toss.)

Suppose that an experiment in which the coin is tossed $N = 1{,}000{,}000$ times is patiently performed. The fraction of tosses resulting in heads is a random variable denoted by p_N. In this particular experiment, suppose that the measured value of p_N is 0.5024.

The coin-tossing experiment is an example of Bernoulli trials, for which there is a complete theory. For example, suppose that we hypothesize that the coin is 'fair,' i.e., $p = \frac{1}{2}$. Then a simple statistical calculation shows that (for $N = 1{,}000{,}000$) with 99% probability p_N lies in the range

$$0.4987 < p_N < 0.5013.$$

Since the measured value $p_N = 0.5024$ lies outside this range, we can have high confidence that the hypothesis $p = \frac{1}{2}$ is false. Instead, a further statistical

calculation, based on the observed value of p_N, shows that, with 99% confidence, p lies in the range

$$0.5011 < p < 0.5037.$$

To summarize:

(i) p is *defined* to be the probability of 'heads,'
(ii) p_N is the *measured* frequency of 'heads,'
(iii) given a hypothesis about p, a *range* for p_N can be predicted, and
(iv) given the measured value of p_N, a *confidence interval* for p can be determined.

The two most important points to appreciate are that p cannot be measured – it can only be estimated with some confidence level; and that, although p_N tends to p as N tends to infinity, this is not taken as the definition of p.

In considering the velocity $U(t)$ as a turbulent flow, we *define* $f(V;t)$ to be its PDF, and then define the mean by

$$\langle U(t)\rangle \equiv \int_{-\infty}^{\infty} V f(V;t)\,\mathrm{d}V. \tag{3.172}$$

In turbulent-flow experiments and simulations, several kinds of *averaging* are used to define other means that can be related to $\langle U(t)\rangle$. For statistically stationary flows the time average (over a time interval T) is defined by

$$\langle U(t)\rangle_T \equiv \frac{1}{T}\int_t^{t+T} U(t')\,\mathrm{d}t'. \tag{3.173}$$

For flows that can be repeated or replicated N times, the ensemble average is defined by

$$\langle U(t)\rangle_N \equiv \frac{1}{N}\sum_{n=1}^{N} U^{(n)}(t), \tag{3.174}$$

where $U^{(n)}(t)$ is the measurement on the nth realization. In simulations of homogeneous turbulence in a cubic domain of side $\mathcal{L}$, the spatial average of $U(\boldsymbol{x},t)$ is defined by

$$\langle U(t)\rangle_{\mathcal{L}} \equiv \frac{1}{\mathcal{L}^3}\int_0^{\mathcal{L}}\int_0^{\mathcal{L}}\int_0^{\mathcal{L}} U(\boldsymbol{x},t)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \tag{3.175}$$

Similar spatial averages can be defined for statistically one- and two-dimensional flows.

These averages $\langle U\rangle_T$, $\langle U\rangle_N$, and $\langle U\rangle_{\mathcal{L}}$ are (like p_N) random variables. They can be used to *estimate* $\langle U\rangle$, but not to measure it with certainty. Most importantly, $\langle U\rangle$ is well defined for all flows, even those that are not stationary

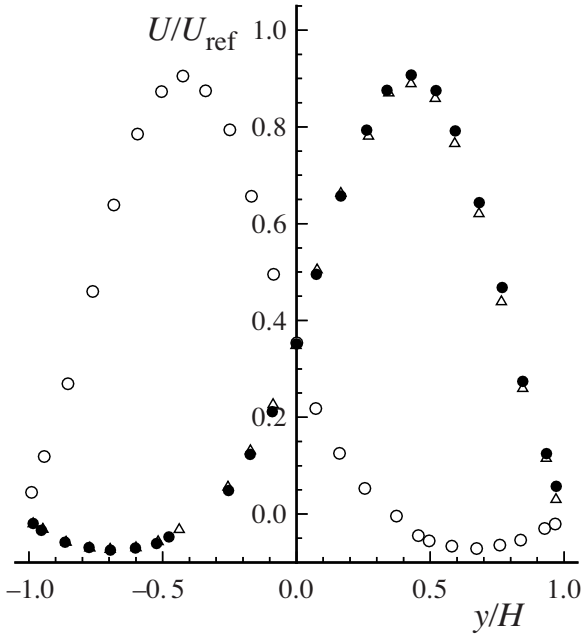

Fig. 3.26. Velocity profiles measured by Durst *et al.* (1974) in the steady laminar flow downstream of a symmetric expansion in a rectangular duct. The geometry and boundary conditions are symmetric about the plane $y = 0$. Symbols: ○, stable state 1; △, stable state 2; •, reflection of profile 1 about the y axis.

or homogeneous, or that cannot be repeated or replicated. For statistically stationary flows (barring exceptional circumstances) $\langle U \rangle_T$ tends to $\langle U \rangle$ as T tends to infinity, but this is not taken as the definition of the mean.

EXERCISES

3.36 In a turbulent-flow experiment the ensemble mean $\langle U \rangle_N$ obtained from $N = 1{,}000$ measurements is 11.24 m s^{-1}, and the standard deviation of U is estimated to be 2.5 m s^{-1}. Determine the 95% confidence interval for $\langle U \rangle$.

3.37 For a statistically stationary flow show that

$$\text{var}(\langle U(t) \rangle_T) = \frac{\text{var}(U)}{T^2} \int_0^T \int_0^T \rho(t - s)\, \text{d}s\, \text{d}t,$$

where $\rho(s)$ is the autocorrelation function of $U(t)$. Assuming that the integral timescale $\bar{\tau}$ exists and is positive (Eq. (3.139)), obtain the long-time result

$$\text{var}(\langle U(t) \rangle_T) \sim \frac{2\bar{\tau}}{T} \text{var}(U). \tag{3.176}$$

3.38 Figure 3.26 shows velocity profiles measured in the steady laminar

flow downstream of a symmetric expansion in a rectangular duct. Although the geometry and boundary conditions are symmetric about the plane $y = 0$, the flow is not symmetric. Each time the flow is started from rest, after an initial transient, the flow reaches one of two stable steady states. For this flow, discuss the relationship among the expectation $\langle U \rangle$, the time average $\langle U \rangle_T$, and the ensemble average $\langle U \rangle_N$.