LES of Turbulent Flows: Lecture 15

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Overview

1. Similarity Models

Similarity Models

- So far we have focused on eddy-viscosity models and now we will look at similarity models (See Saguat pg 231 for examples)
- Scale similarity assumes that the statistical structure of tensors constructed on the basis of the SFS is similar to that of their equivalents evaluated on the basis of the smallest resolved scales
- Accordingly, the spectrum is usually separated into three bands
  - the largest resolved scales
  - the smallest resolved scales (also called the test field)
  - the unresolved scales

Similarity Models

- The notion of scale similarity models can be interpreted in two ways
- One relates to energy cascade, with the idea that the unresolved scales and smallest resolved scales have a common history through interactions with largest scales
- The other relates to coherent structures, where some structures appear in each of the three bands and cause a strong correlation of the field among each level of decomposition

Similarity Models

- Bardina et al., (1980) proposed an alternative model to the eddy-viscosity model
- They authors were motivated by the low correlations between $\tau_{ij}^{\Delta}(\vec{x},t)$ and $\tau_{ij}^{\Delta,M}(\vec{x},t)$ in *a priori* studies
- We will cover *a priori* studies in a later lecture
- The subgrid stress tensor is found by applying the filter a second time, which is a means to evaluate the fluctuation of the resolved scales
- As a result, this model cannot be used when the filter is idempotent because the fluctuation is zero

Similarity Models

- Recall that the SFS velocity is defined as $u_i' = u_i - \widetilde{u}_i$ and the filtered SFS velocity is $\widetilde{u_i'} = \widetilde{u}_i - \widetilde{\widetilde{u}}_i$
- Also recall Leonard's decomposition of τ_{ij}

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij}$$

where

$$L_{ij} = \underbrace{\left(\widetilde{\widetilde{u}_i\widetilde{u}_j} - \widetilde{u}_i\widetilde{u}_j\right)}_{\text{Resolved stresses}}$$

$$C_{ij} = \underbrace{\left(\widetilde{\widetilde{u}_i u_j'} + \widetilde{u_i'\widetilde{u}_j}\right)}_{\text{Cross stresses}}$$

$$R_{ij} = \underbrace{\widetilde{u_i' u_j'}}_{\text{``Reynolds'' stresses}}$$

Similarity Models

- Using our definition of the filtered velocity fluctuations, and the following assumption shown for R_{ij}, we can write each of our terms as follows

$$
\begin{aligned}
R_{ij} &= \widetilde{\left(u_i - \widetilde{u}_i\right)\left(u_j - \widetilde{u}_j\right)} \\
&\approx \left(\widetilde{u}_i - \widetilde{\widetilde{u}}_i\right)\left(\widetilde{u}_j - \widetilde{\widetilde{u}}_j\right) \\
C_{ij} &\approx \widetilde{\widetilde{u}}_i \left(\widetilde{u}_j - \widetilde{\widetilde{u}}_j\right) + \widetilde{\widetilde{u}}_j \left(\widetilde{u}_i - \widetilde{\widetilde{u}}_i\right) \\
L_{ij} &= \left(\widetilde{\widetilde{u}_i \widetilde{u}_j} - \widetilde{u}_i \widetilde{u}_j\right)
\end{aligned}
$$

Similarity Models

- Let's add them up and do a little algebra

$$
\begin{aligned}
\tau_{ij} = &\left(\widetilde{\tilde{u}_i\tilde{u}_j} - \tilde{u}_i\tilde{u}_j\right) + \tilde{\tilde{u}}_i\left(\tilde{u}_j - \tilde{\tilde{u}}_j\right) + \tilde{\tilde{u}}_j\left(\tilde{u}_i - \tilde{\tilde{u}}_i\right) \\
&+ \left(\tilde{u}_i - \tilde{\tilde{u}}_i\right)\left(\tilde{u}_j - \tilde{\tilde{u}}_j\right) \\
=&\tilde{u}_i\tilde{u}_j - \tilde{u}_i\tilde{\tilde{u}}_j - \tilde{\tilde{u}}_i\tilde{u}_j + \tilde{\tilde{u}}_i\tilde{\tilde{u}}_j + \tilde{\tilde{u}}_i\tilde{u}_j - \tilde{\tilde{u}}_i\tilde{\tilde{u}}_j + \tilde{\tilde{u}}_j\tilde{u}_i \\
&- \tilde{\tilde{u}}_j\tilde{\tilde{u}}_i + \left(\widetilde{\tilde{u}_i\tilde{u}_j} - \tilde{u}_i\tilde{u}_j\right)
\end{aligned}
$$

Simple elimination yields

$$
\boxed{\tau_{ij} = \left(\widetilde{\tilde{u}_i\tilde{u}_j} - \tilde{\tilde{u}}_i\tilde{\tilde{u}}_j\right)}
$$

which gives an estimate for the SGS stress

Similarity Models

- The Bardina model does not require physical modeling of SFSs, rather it is a mathematical approximation of τ_{ij}
- *a priori* tests against DNS databases showed that the Bardina model performed well
- The model produced high correlations with the true subgrid stress tensor
- These correlations occurred for both isotropic and anisotropic flows
- However, results also showed that the model is only slightly dissipative and underestimates the energy cascade
- The model also has a built in backscatter mechanism

- The Bardina model applied the same filter twice, meaning it used a single cutoff scale
- Liu et al. (1994) generalized the Bardina model to allow the use of filters with different shapes and widths, which allows it to be used for any type of filter.
- The authors examined "bands" around Δ and built a scale-similarity model similar to the model of Bardina et al. (1980)
- They argued that energy in the band at one scale larger than Δ (say 2Δ) and one scale smaller (something like 0.5Δ) would have the largest contribution to τ_{ij}.

Similarity Models

- Define $u_i^n = \widetilde{u}_i - \bar{u}_i$, where $(\,\widetilde{\ }\,)$ is a filter at Δ and $(\,\bar{\ }\,)$ is a filter at a larger scale 2Δ
- We can do a similar decomposition for u_i^{n+1} and u_i^{n-1}
- With our band-pass filtered decomposition, we can build a τ_{ij}^n based on u_i^n and u_i^{n+1} (or any other band)

Similarity Models

- For example, the stress one level above n can be written using another filter at 4Δ, denoted by $(\,\hat{}\,)$, as

$$\tau_{ij}^{n-1} = \overline{(\widetilde{u}_i - \hat{u}_i)(\widetilde{u}_j - \hat{u}_j)} - \overline{(\widetilde{u}_i - \hat{u}_i)}\;\overline{(\widetilde{u}_j - \hat{u}_j)}$$
$$= \overline{\widetilde{u}_i\widetilde{u}_j - \widetilde{u}_i\hat{u}_j - \hat{u}_i\widetilde{u}_j + \hat{u}_i\hat{u}_j} - \overline{(\widetilde{u}_i - \hat{u}_i)}\;\overline{(\widetilde{u}_j - \hat{u}_j)}$$
$$= \overline{\widetilde{u}_i\widetilde{u}_j} - \overline{\widetilde{u}_i}\hat{u}_j - \hat{u}_i\overline{\widetilde{u}_j} + \hat{u}_i\hat{u}_j - \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j} + \overline{\widetilde{u}_i}\hat{u}_j + \hat{u}_i\overline{\widetilde{u}_j} - \hat{u}_i\hat{u}_j$$

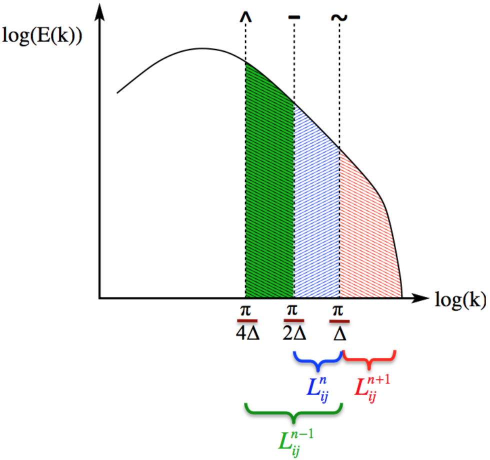
$$\boxed{\tau_{ij}^{n-1} = \left(\overline{\widetilde{u}_i\widetilde{u}_j} - \overline{\widetilde{u}_i}\;\overline{\widetilde{u}_j}\right)}$$

Note: $\hat{u}_i$ is approximately a constant with respect to the $(\,\bar{}\,)$ filter

Similarity Models

- Liu et al. (1994) study showed similarity between

$$\underbrace{\tau_{ij}^{n+1}}_{1^{\text{st}}\text{ unresolved band}} \longrightarrow \underbrace{\tau_{ij}^{n}}_{\text{smallest resolved band}} \longrightarrow \underbrace{\tau_{ij}^{n-1}}_{\text{next largest resolved band}}$$



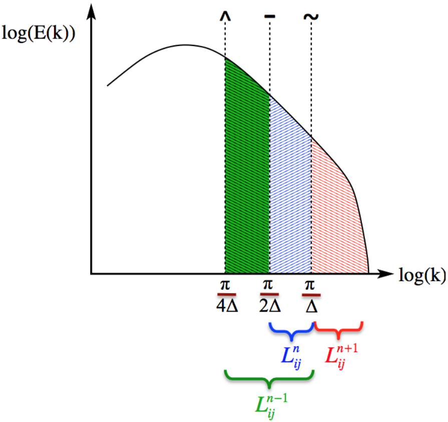
Similarity Models

- They concluded that because of this the Leonard stress (τ_{ij}^{n-1}) is the best estimate

$$\boxed{\tau_{ij} = C_L L_{ij}}$$

where $L_{ij} = \left(\overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\,\overline{\widetilde{u}_j}\right)$ and $C_L \sim 1$ is a dimensionless coefficient

- This is the most commonly used form (currently) of the similarity model



Similarity Models

- The Bardina and Liu models lead to high correlations between $\tau_{ij}^{\Delta}(\vec{x},t)$ and $\tau_{ij}^{\Delta,M}(\vec{x},t)$
- However, these models are expensive computationally due to the application of multiple explicit filtering operations
- Another procedure was introduced that reduces this cost, called the nonlinear model (a.k.a. Clark model, gradient model, or tensor-diffusivity model)

Similarity Models

- The idea behind the nonlinear model is to approximate $\widetilde{u}_i$ by a Taylor series expansion around the "true" mean at a point

$$\widetilde{u}_i(\vec{x}) = \overline{\widetilde{u}_i} + \widetilde{A}_{ijk}(\vec{x_0})(x_k - x_k^0)$$

where $\widetilde{A}_{ijk}$ is the filtered gradient tensor, given by

$$\widetilde{A}_{ijk} = \frac{\partial \widetilde{u}_i}{\partial x_k}$$

Similarity Models

- We can use this approximation (Taylor series) to estimate the "resolved" stress $L_{ij} = \overline{\widetilde{u}_i \widetilde{u}_j} - \overline{\widetilde{u}_i}\, \overline{\widetilde{u}_j}$ (more later during discussion of dynamic modeling) and develop another model

$$\boxed{\tau_{ij} = C_A \Delta^2 \widetilde{A}_{ik} \widetilde{A}_{jk}}$$

- Here we have used the observation that τ_{ij} has a very high correlation with L_{ij}

$$\Rightarrow \tau_{ij} = C_A L_{ij}$$

- See Sagaut pg 231-231 for an example derivation

Similarity Models

- The model shares the same order of deviation from the actual τ_{ij} as the Bardina-type models
- The primary advantage is that no additional explicit filtering operations are required
- As a result, the model is far less computationally expensive