LES of Turbulent Flows: Lecture 13

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Overview

1. Recap: Eddy-Viscosity Models
2. Smagorinsky Model

Eddy-Viscosity Models

- In Lecture 12 we said that eddy-viscosity models are of the form:

$$\text{momentum} \qquad \tau_{ij} = -2\nu_T \widetilde{S}_{ij}$$

$$\text{scalars} \qquad q_i = -D_T \frac{\partial \widetilde{\theta}}{\partial x_i}$$

where

$$D_T = \frac{\nu_T}{\mathsf{Pr}},$$

- $\widetilde{S}_{ij}$ is the filtered strain rate
- ν_T is eddy-viscosity
- D_T is eddy-diffusivity
- Pr is the SGS Prandtl number

Eddy-Viscosity Models

- This is the LES equivalent to 1st-order RANS closure (k-theory or gradient transport theory) and is an analogy to molecular viscosity
- Turbulent fluxes are assumed to be proportional to the local velocity or scalar gradients
- In LES, this is the assumption that stress is proportional to strain: $\tau_{ij} \sim \widetilde{S}_{ij}$
- The SGS eddy-viscosity ν_T must be parameterized

Eddy-Viscosity Models

- Dimensionally

$$\nu_T = \left[\frac{L^2}{T}\right]$$

- In almost all SGS eddy-viscosity models:

$$\nu_T \sim U\ell$$

- Different models use different U and ℓ

Eddy-Viscosity Models

Interpretation

- Recall that we can interpret the eddy-viscosity as adding to the molecular viscosity.

$$2\frac{\partial}{\partial x_j}\left[\left(\nu_T+\nu\right)\widetilde{S}_{ij}\right]$$

What does the model do?

- We can see it effectively lowers the Reynolds number of the flow
- It provides all of the energy dissipation for high Re flows (when 1/Re$\Rightarrow$0).

Smagorinsky Model

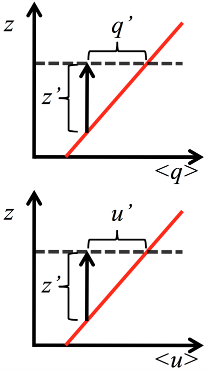
- One of the first and most popular eddy-viscosity models for LES is the Smagorinsky model (Smagorinsky 1963)
- The model was originally developed for general circulation (large-scale atmospheric) models
- The model did not remove enough energy in this context

Smagorinsky Model

- The Smagorinsky model was applied by Deardorff (1970) in the first reported LES
- Deardorff used Prandtl's mixing length idea (1925) applied at the SGSs (see Pope Ch. 10 or Stull, 1988 for a full review of mixing length)

Smagorinsky Model

Prandtl's mixing length – for a general scalar quantity q with an assumed linear profile:



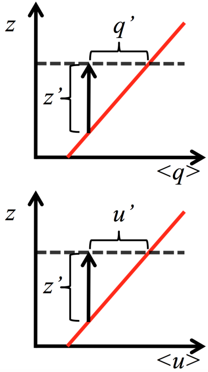
- A turbulent eddy moves a parcel of air by an amount z' towards a level z where we have no mixing or other change
- q will differ from the surrounding air by:

$$q' = -\left(\frac{\partial \langle q \rangle}{\partial z}\right) z'$$

i.e., the scalar will change proportional to its local gradient

Smagorinsky Model

Prandtl's mixing length – likewise, for velocity u with an assumed linear profile:



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i.e., the velocity will change proportional to its local gradient

Smagorinsky Model

Prandtl's mixing length

- In order to move up a distance z', our air parcel must have some vertical velocity w'
- If turbulence is such that $w' \propto u'$, then $w' = Cu'$
- We have two cases:

$$\frac{\partial u}{\partial z} > 0 \Rightarrow w' = -Cu'$$

$$\frac{\partial u}{\partial z} < 0 \Rightarrow w' = Cu'$$

- Combining these, we get that:

$$w' = C\left|\frac{\partial \langle u\rangle}{\partial z}\right| z'$$

Smagorinsky Model

Prandtl's mixing length

- We now have:

$$q' = -\left(\frac{\partial \langle q \rangle}{\partial z}\right) z'$$
$$w' = C \left| \frac{\partial \langle u \rangle}{\partial z} \right| z'$$

- We can form a kinematic flux (conc. * velocity) by multiplying the two together:

$$\langle w' q' \rangle = -C \langle (z')^2 \rangle \left| \frac{\partial \langle u \rangle}{\partial z} \right| \frac{\partial \langle q \rangle}{\partial z}$$

Smagorinsky Model

Prandtl's mixing length

- Prandtl assumed that the constant of proportionality is unity and called z' the mixing length (ℓ)

$$\langle w'q' \rangle = -\ell^2 \left| \frac{\partial \langle u \rangle}{\partial z} \right| \frac{\partial \langle q \rangle}{\partial z}$$

- We can replace q' with any variable in this relationship (*e.g.*, u')
- You can think of $(z')^2$ or ℓ^2 as the variance of a parcel's movement

Smagorinsky Model

Prandtl's mixing length - what is it? Let's ask Prandtl.

> *may be considered as the diameter of the masses of fluid moving as a whole in each individual case; or again, as the distance traversed by a mass of this type before it becomes blended in with neighbouring masses*
>
> *...*
>
> *somewhat similar, as regards effect, to the mean free path in the kinetic theory of gases*
>
> *...*
>
> *only a rough approximation*

In other words, the mixing length according to Prandtl is the average distance that a fluid mass will travel before it is changed by the new environment

Back to the Smagorinsky model

Recall:

- Dimensionally

$$\nu_T = \left[\frac{L^2}{T}\right]$$

- In almost all SGS eddy-viscosity models:

$$\nu_T \sim \ell U$$

- Different models use different ℓ and U

- Use Prandtl's mixing length applied at the SGSs

$$\nu_T = \underbrace{(C_S \Delta)^2 |\widetilde{S}|}_{\ell \ell T^{-1} = \ell U}$$

Smagorinsky Model

$$\nu_T = (C_S \Delta)^2 |\widetilde{S}|$$

- $\Delta = (\Delta_x \Delta_y \Delta_z)^{\frac{1}{3}}$ is the effective grid scale (see Deardorff 1970 or Scotti et al, 1993)
- $C_S \Delta$ is the length scale – squared for dimensional consistency
- $|\widetilde{S}| = \sqrt{2\widetilde{S}_{ij}\widetilde{S}_{ij}}$ is the magnitude of the filtered strain rate tensor with units of $[T^{-1}]$. It serves as part of the velocity scale – think $\partial\langle u\rangle/\partial z$ in Prandtl's theory

Smagorinsky Model

- The final model is

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\nu_T \widetilde{S}_{ij} = -2(C_S\Delta)^2|\widetilde{S}|\widetilde{S}_{ij}$$

- In order to close the model, we need a value of C_S (usually called the Smagorinsky or Smagorinsky-Lilly coefficient)

Smagorinsky Model

- Lilly (1967) proposed a method to determine C_S (see also Pope pg 587)
- We assume that we have a high-Re flow such that Δ can be taken to be in the inertial subrange of turbulence
- The mean energy transfer across Δ must be balanced, on average, by viscous dissipation (note: for Δ in the inertial subrange this is not an assumption)

$$\epsilon = \langle \Pi \rangle$$

recall: $$\Pi = -\tau_{ij} \widetilde{S}_{ij}$$

Smagorinsky Model

- Using an eddy-viscosity model, we get

$$\Pi = -\tau_{ij}\widetilde{S}_{ij} = 2\nu_T\widetilde{S}_{ij}\widetilde{S}_{ij} = \nu_T|\widetilde{S}|^2$$

- We can use the Smagorinsky model

$$\nu_T = (C_S\Delta)^2|\widetilde{S}|$$

to arrive at:

$$\Pi = (C_S\Delta)^2|\widetilde{S}|^3$$

Smagorinsky Model

- The square of $|\widetilde{S}|$ can be written as (see Liily 1967 and Pope pg 579):

$$|\widetilde{S}|^2 = 2\int_0^\infty k^2 \hat{G}(k)^2 E(k) dk$$

  where $\hat(G)(k)$ is the filter transfer function and $E(k)$ is the energy spectrum

- Recall that for a Kolmogorov spectrum in the inertial subrange:

$$E(k) \sim C_k \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

Smagorinsky Model

- Substitution of the Kolmogorov $E(k)$ yields

$$
\begin{aligned}
|\widetilde{S}|^2 &= 2\int_0^\infty k^2 \hat{G}(k)^2 E(k)dk \approx 2\int_0^\infty k^2 \hat{G}(k)^2 C_k \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} dk \\
&\approx a_f C_k \epsilon^{\frac{2}{3}} \Delta^{-\frac{4}{3}}
\end{aligned}
$$

where

$$
a_f = 2\int_0^\infty (k\Delta)^{\frac{1}{3}} \hat{G}(k)^2 \Delta dk
$$

Note that a_f depends on the filter, but is independent of Δ

Smagorinsky Model

- We can rearrange $|\widetilde{S}|^2 = a_f C_k \epsilon^{\frac{2}{3}} \Delta^{-\frac{4}{3}}$ to get:

$$\epsilon = \left[\frac{\langle |\widetilde{S}|^2 \rangle}{a_f C_k \Delta^{-\frac{4}{3}}} \right]^{\frac{3}{2}}$$

- Recall that we are equating viscous dissipation and the average Smagorinsky SGS dissipation:

$$\epsilon = \langle \Pi \rangle$$

where

$$\Pi = (C_S \Delta)^2 |\widetilde{S}|^3$$

Smagorinsky Model

- Combining these expression yields:

$$(C_S\Delta)^2\langle|\widetilde{S}|^3\rangle = \left[\frac{\langle|\widetilde{S}|^2\rangle}{a_f C_k \Delta^{-\frac{4}{3}}}\right]^{\frac{3}{2}}$$

  Algebra then gives us:

$$C_S = \frac{1}{(a_f C_k)^{3/4}}\left(\frac{\langle|\widetilde{S}_{ij}|^3\rangle}{\langle|\widetilde{S}_{ij}|^2\rangle^{3/2}}\right)^{-\frac{1}{2}}$$

Smagorinsky Model

- We can approximate that $\langle|\widetilde{S}_{ij}|^2\rangle^{3/2} \approx \langle|\widetilde{S}_{ij}|^3\rangle$
- We can also make use of $a_f = (3/2\pi)^{4/3}$ for a sharp cutoff filter (see Pope)
- Substitution yields

$$C_S = \frac{1}{\pi}\left(\frac{2}{3C_k}\right)^{\frac{3}{4}}$$

- C_k is the Kolmogorov “constant” and is generally taken as $C_k \approx 1.5$-1.6. Using this values leads to

$$\boxed{C_S \approx 0.17}$$