LES of Turbulent Flows: Lecture 6

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Overview

1. Recap of filters
2. Deriving the incompressible equations of motion
3. Non-dimensional incompressible equations of motion
4. Filtering the incompressible equations of motion

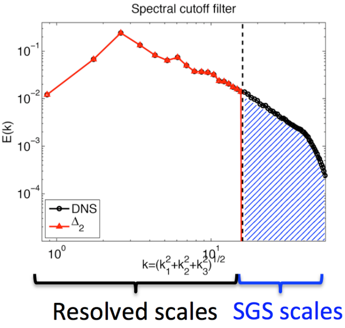
Decomposition of turbulence for real filters

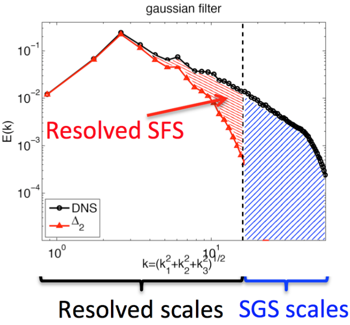
- The LES filter can be used to decompose the velocity field into resolved and subfilter scale (SFS) components:

$$\underbrace{\phi(\vec{x},t)}_{\text{total}} = \underbrace{\tilde{\phi}(\vec{x},t)}_{\text{resolved}} + \underbrace{\phi'(\vec{x},t)}_{\text{subfilter}}$$

- We can use our filtered DNS fields to look at how the choice of our filter kernel affects this separation in wavespace.

Decomposition of turbulence for real filters





- The Gaussian (or box) filter does not have as compact of support in wavespace as the cutoff filter.
- This results in attenuation of energy at scales larger than the filter scale.
- The scales affected by the attenuation are referred to as *resolved SFSs*.

Deriving the incompressible equations of motion

- We want to apply the filters to the N-S equations of motion.
- First, let's start with the fully compressible form of the equations of motion and derive the incompressible counterparts.
- Next, we will apply the filtering operation to the incompressible equations of motion.
- Lastly, we will relate the final forms of the equations to the conceptual idea of LES.

Conservation of mass

We start with the full equation for the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0$$

We apply the incompressibility condition – that a fluid parcel's density is constant ($\rho = \rho_o$):

$$\cancelto{0}{\frac{\partial \rho_o}{\partial t}} + \cancel{\rho_o}\frac{\partial u_i}{\partial x_i} = 0$$

Finally, we divide by ρ_o to arrive at the conservation of mass equation for incompressible flows:

$$\boxed{\frac{\partial u_i}{\partial x_i} = 0}$$

Conservation of momentum

We start with the full equation for the conservation of momentum:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right] - \frac{\partial p}{\partial x_j} + F_i$$

Apply the incompressibility condition:

$$\rho_o\frac{\partial u_i}{\partial t} + \rho_o\frac{\partial(u_i u_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right] - \frac{\partial p}{\partial x_j} + F_i$$

Divide by ρ_o:

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_j)}{\partial x_j} = \frac{\mu}{\rho_o}\frac{\partial}{\partial x_j}\left[2S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right] - \frac{1}{\rho_o}\frac{\partial p}{\partial x_j} + F_i$$

Conservation of momentum

Recall that

$$\nu = \mu/\rho_o \qquad \text{and} \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

To arrive at:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \nu \frac{\partial}{\partial x_j}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\mu\delta_{ij}\cancelto{0}{\frac{\partial u_i}{\partial x_i}}\right] - \frac{1}{\rho_o}\frac{\partial p}{\partial x_j} + F_i$$

And we can apply the incompressible mass conservation equation and distribute the $\partial/\partial x_j$ in the first term on the right side:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \nu\left[\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_i}{\partial x_i \partial x_j}\right] - \frac{1}{\rho_o}\frac{\partial p}{\partial x_j} + F_i$$

Conservation of momentum

We can rearrange and again apply the mass continuity equation:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \nu \frac{\partial^2 u_i}{\partial x_j^2} + \nu \frac{\partial}{\partial x_j} \cancelto{0}{\left(\frac{\partial u_i}{\partial x_i} \right)} - \frac{1}{\rho_o} \frac{\partial p}{\partial x_j} + F_i$$

and we arrive at the conservation of momentum equation for incompressible flows:

$$\boxed{\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i}$$

Conservation of a general scalar

We start with the full equation for the conservation of momentum:

$$\frac{\partial(\rho\theta)}{\partial t} + \frac{\partial(\rho u_i \theta)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\nu_\theta \rho \frac{\partial \theta}{\partial x_j}\right] + Q$$

Apply the incompressibility condition:

$$\cancel{\rho_o}\frac{\partial\theta}{\partial t} + \cancel{\rho_o}\frac{\partial(u_i \theta)}{\partial x_j} = \nu_\theta \cancel{\rho_o}\frac{\partial}{\partial x_j}\left[\frac{\partial \theta}{\partial x_j}\right] + Q$$

Divide by ρ_o and we arrive at the conservation of a general scalar equation for incompressible flows:

$$\boxed{\frac{\partial\theta}{\partial t} + \frac{\partial(u_i \theta)}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} + Q}$$

Non-dimensional incompressible equations of motion

Recall that we can non-dimensionalize these equations by using representative scales, U and ℓ:

$$u_i^* = \frac{u_i}{U}$$

$$x_i^* = \frac{x_i}{\ell}$$

$$p^* = \frac{p}{\rho U^2}$$

$$t^* = \frac{tU}{\ell}$$

$$\theta^* = \frac{\theta}{\theta_o}$$

where the * denotes a non-dimensionalized term.

Non-dimensional conservation of mass

Start with the incompressible conservation of mass and apply the non-dimensional relationships:

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial (u_i^* U)}{\partial (x_i^* \ell)} = \frac{\cancel{U}}{\cancel{\ell}} \frac{\partial u_i^*}{\partial x_i^*} = 0$$

divide by U/ℓ to arrive at the non-dimensional incompressible conservation of mass:

$$\boxed{\frac{\partial u_i^*}{\partial x_i^*} = 0}$$

Non-dimensional conservation of momentum

Start with the incompressible conservation of momentum and apply the non-dimensional relationships:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i$$

$$\Rightarrow \frac{\partial (u_i^* U^2)}{\partial t^* \ell} + \frac{\partial (u_i^* u_j^* U^2)}{\partial x_j^* \ell} = -\frac{1}{\rho_o}\frac{\partial (p^* \rho_o U^2)}{\partial x_j^* \ell} + \nu \frac{\partial^2 (u_i^* U)}{\partial x_j^{*2} \ell^2} + F_i$$

Recall that Re$=U\ell/\nu \Rightarrow \nu = U\ell/$Re:

$$\cancel{\frac{U^2}{\ell}}\frac{\partial u_i^*}{\partial t^*} + \cancel{\frac{U^2}{\ell}}\frac{\partial (u_i^* u_j^*)}{\partial x_j^*} = -\cancel{\frac{U^2}{\ell}}\frac{1}{\cancel{\rho_o}}\frac{\partial (p^* \cancel{\rho_o})}{\partial x_j^*} + \cancel{\frac{U^2}{\ell}}\frac{1}{Re}\frac{\partial^2 u_i^*}{\partial x_j^{*2}} + F_i$$

divide by U^2/ℓ to arrive at the non-dimensional incompressible conservation of momentum:

$$\boxed{\frac{\partial u_i^*}{\partial t^*} + \frac{\partial (u_i^* u_j^*)}{\partial x_j^*} = -\frac{\partial p^*}{\partial x_j^*} + \frac{1}{Re}\frac{\partial^2 u_i^*}{\partial x_j^{*2}} + F_i}$$

Non-dimensional conservation of a general scalar

Start with the incompressible conservation of a general scalar and apply the non-dimensional relationships:

$$\frac{\partial \theta}{\partial t} + \frac{\partial (u_i \theta)}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} + Q$$

$$\Rightarrow \frac{\partial \theta^* \theta_o U}{\partial t^* \ell} + \frac{\partial (u_i^* \theta^* \theta_o)}{\partial x_j^* \ell} = \nu_\theta \frac{\partial^2 \theta^* \theta_o}{\partial x_j^{*2} \ell^2} + Q$$

Recall that Sc$= \nu/\nu_\theta \Rightarrow \nu_\theta = \nu/Sc= U\ell/($Sc Re$)$:

$$\cancel{\frac{U\theta_o}{\ell}} \frac{\partial \theta^*}{\partial t^*} + \cancel{\frac{U\theta_o}{\ell}} \frac{\partial (u_i^* \theta^*)}{\partial x_j^*} = \cancel{\frac{U\theta_o}{\ell}} \frac{1}{Sc\ Re} \frac{\partial^2 \theta^*}{\partial x_j^{*2}} + Q$$

divide by $U\theta_o/\ell$ to arrive at the non-dimensional incompressible conservation of a general scalar:

$$\boxed{\frac{\partial \theta^*}{\partial t^*} + \frac{\partial (u_i^* \theta^*)}{\partial x_j^*} = + \frac{1}{Sc\ Re} \frac{\partial^2 \theta^*}{\partial x_j^{*2}} + Q}$$

Filtering the incompressible equations of motion

Next we apply the filter to the non-dimensional incompressible equations of motion, recalling that the filters hold the following properties:

$$\tilde{a} = a$$
$$\widetilde{\phi + \zeta} = \tilde{\phi} + \tilde{\zeta}$$
$$\widetilde{\frac{\partial \phi}{\partial x}} = \frac{\partial \tilde{\phi}}{\partial x}$$

That is: a constant is unaffected by the filter, the filtered sum of two variables is the sum of the filtered variables, and the filter is commutative for differentiation.

Filtered conservation of mass

Start with the non-dimensional incompressible conservation of mass and apply the filter (where the * notation is dropped for convenience):

$$\widetilde{\frac{\partial u_i}{\partial x_i} = 0}$$

$$\widetilde{\frac{\partial u_i}{\partial x_i}} = \tilde{0}$$

$$\boxed{\frac{\partial \tilde{u}_i}{\partial x_i} = 0}$$

This is the non-dimensional form of the filtered conservation of mass equation for incompressible flows.

Filtered conservation of momentum

Start with the non-dimensional incompressible conservation of momentum and apply the filter (where the * notation is dropped for convenience):

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} &= \widetilde{-\frac{\partial p}{\partial x_j}} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2} + F_i \\
\widetilde{\frac{\partial u_i}{\partial t}} + \widetilde{\frac{\partial (u_i u_j)}{\partial x_j}} &= \widetilde{-\frac{\partial p}{\partial x_j}} + \widetilde{\frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2}} + F_i \\
\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial (\widetilde{u_i u_j})}{\partial x_j} &= -\frac{\partial \tilde{p}}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} + F_i
\end{aligned}
$$

We have a problem because $\widetilde{u_i u_j}$ is the filtered product of two non-filtered variables. We do not have knowledge of these variables and thus the term cannot be solved *a priori*.

Filtered conservation of momentum

Following Leonard (1974), we can decompose the unknown term as

$$\widetilde{u_i u_j} = \tilde{u}_i \tilde{u}_j + \tau_{ij}^r$$

where τ_{ij}^r is the subfilter scale (SFS) stress tensor.

We can substitute this back into the previous equation to arrive at the non-dimensional form of the filtered conservation of momentum equation for incompressible flows:

$$\boxed{\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial (\tilde{u}_i \tilde{u}_j)}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}^r}{\partial x_j} + F_i}$$

Welcome to the closure problem because τ_{ij}^r is unknown – thus the equation is not closed. The SFS stress tensor must be modeled.

Filtered conservation of a general scalar

Start with the non-dimensional incompressible conservation of a general scalar and apply the filter (where the * notation is dropped for convenience):

$$\widetilde{\frac{\partial \theta}{\partial t} + \frac{\partial (u_i \theta)}{\partial x_j} = + \frac{1}{Sc\ Re} \frac{\partial^2 \theta}{\partial x_j^2} + Q}$$

$$\widetilde{\frac{\partial \theta}{\partial t}} + \widetilde{\frac{\partial (u_i \theta)}{\partial x_j}} = + \widetilde{\frac{1}{Sc\ Re} \frac{\partial^2 \theta}{\partial x_j^2}} + Q$$

$$\frac{\partial \tilde{\theta}}{\partial t} + \frac{\partial (\widetilde{u_i \theta})}{\partial x_j} = \frac{1}{Sc\ Re} \frac{\partial^2 \tilde{\theta}}{\partial x_j^2} + Q$$

Again, we have a problem because $\widetilde{u_i \theta}$ is the filtered product of two non-filtered variables. We do not have knowledge of these variables and thus the term cannot be solved *a priori*.

Filtered conservation of a general scalar

We again decompose the unknown term as

$$\widetilde{u_i\theta} = \tilde{u}_i\tilde{\theta} + q_i^r$$

where q_i^r is the SFS flux.

We can substitute this back into the previous equation to arrive at the non-dimensional form of the filtered conservation of momentum equation for incompressible flows:

$$\boxed{\frac{\partial\tilde{\theta}}{\partial t} + \frac{\partial(\tilde{u}_i\tilde{\theta})}{\partial x_j} = \frac{1}{Sc\ Re}\frac{\partial^2\tilde{\theta}}{\partial x_j^2} - \frac{\partial q_i^r}{\partial x_j} + Q}$$

Similarly, q_i^r is unknown – thus the equation is not closed. The SFS flux must be modeled.

LES filtered equations for incompressible flows

Mass

- $\frac{\partial \tilde{u}_i}{\partial x_i} = 0$

Momentum

- $\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial (\tilde{u}_i \tilde{u}_j)}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}^r}{\partial x_j} + F_i$

Scalar

- $\frac{\partial \tilde{\theta}}{\partial t} + \frac{\partial (\tilde{u}_i \tilde{\theta})}{\partial x_j} = \frac{1}{Sc\ Re}\frac{\partial^2 \tilde{\theta}}{\partial x_j^2} - \frac{\partial q_i^r}{\partial x_j} + Q$

SFS stress

- $\tau_{ij}^r = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j$

SFS flux

- $q_i^r = \widetilde{u_i \theta} - \tilde{u}_i \tilde{\theta}$

Up next, turbulence kinetic energy

- We've talked about variance (or energy) when discussing turbulence and filtering.
- When we examined the application of the LES filter at scale Δ we saw the effect of the filter on the distribution of energy with scale.
- A natural way to extend our examination of scale separation and energy is to look at the evolution of the filtered variance or turbulence kinetic energy.