LES of Turbulent Flows: Lecture 4

Dr. Jeremy A. Gibbs

Department of Mechanical Engineering
University of Utah

Fall 2016

Overview

1. One last review of turbulence (beat it into your heads)

2. Numerical simulations

3. Equations of motion

Review: Properties of turbulent flows

- Unsteadiness: $u = f(\vec{x}, t)$
- Three-dimensional: $\vec{x} = f(x_i)$ for any turbulent flow
- High vorticity: $\omega = \nabla \times \vec{u}$
- Mixing effect: turbulence acts to reduce gradients
- Continuous spectrum of scales: energy cascade described broadly by Kolmogorov's hypotheses

Review: Kolmogorov's similarity hypothesis

Kolmogorov's 1$^{\text{st}}$ hypothesis

- Smallests scales receive energy at a rate proportional to the dissipation of energy rate
- With this, he defined the Kolmogorov (dissipation) scales:

length scale

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$$

time scale

$$\tau = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}$$

velocity scale

$$v = \frac{\eta}{\nu} = (\nu\epsilon)^{\frac{1}{4}}$$

Review: Kolmogorov's similarity hypothesis

- Using these scales, we can define the ratios of the largest to smallest scales:

length scale

$$\frac{\ell_o}{\eta} \sim \mathsf{Re}^{\frac{3}{4}}$$

velocity scale

$$\frac{U_0}{v} \sim \mathsf{Re}^{\frac{1}{4}}$$

time scale

$$\frac{t_o}{\tau} \sim \mathsf{Re}^{\frac{1}{2}}$$

Review: Kolmogorov's similarity hypothesis

Kolmogorov's 2nd hypothesis

- In turbulent flow, a range of scales exists at very high Re where statistics of motion in a range l ($\ell_o \gg \ell \gg \eta$) have a universal form that is determined only by ϵ (dissipation) and independent of ν (kinematic viscosity).
- Kolmogorov formed his hypothesis and examined it by looking at the PDF of velocity increments Δu.

Review: Kolmogorov's similarity hypothesis

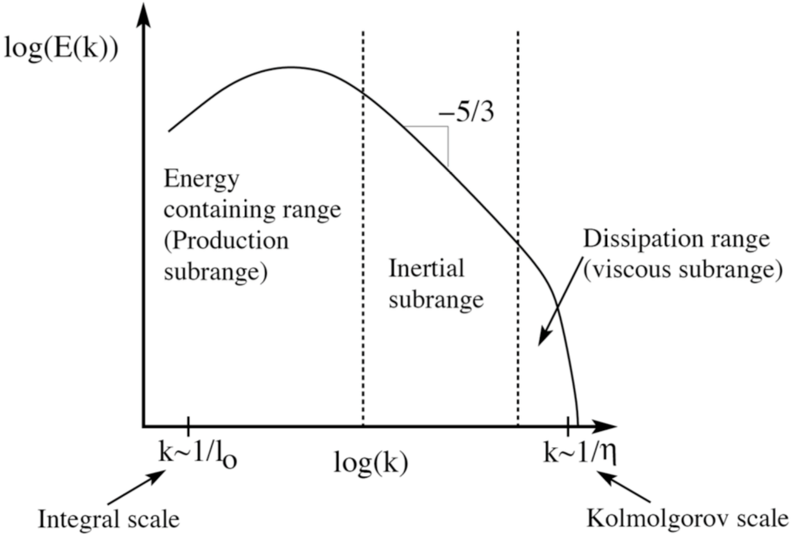
- We can examine this through $E(k)$, where $E(k)dk =$TKE contained between k and $k + dk$.
- What are the implications of Kolmogorov's hypothesis for $E(k)$? – K41$\Rightarrow$ $E(k) = f(k, \epsilon)$
- By dimensional analysis, Kolmogorov showed:

$$E(K) = c_k \epsilon^{2/3} k^{-5/3}$$

Kolmogorov's -5/3 power law.

Review: Kolmogorov's similarity hypothesis

Example energy spectrum



Degrees of freedom and numerical simulations

- We now have a description of turbulence and the range of energy containing scales (the dynamic range) in turbulence.
- In computational fluid dynamics (CFD), we need to discretize the equations of motion using either difference approximations (finite differences) or as a finite number of basis functions (e.g., Fourier transforms).
- Essentially, a continuous solution is approximated by a finite set of values corresponding as closely as possible with the values of the solution on a grid of discrete positions in space.

Degrees of freedom and numerical simulations

- To capture all of the dynamics (degrees of freedom) in a turbulent flow, we must consider the required amount of discrete values needed for an accurate approximation.
- We need a grid fine enough to capture the smallest *and* the largest scales of motion (η and ℓ_o).

Degrees of freedom and numerical simulations

- From K41, we know that $\ell_o/\eta \sim \mathsf{Re}^{3/4}$ and there exists a continuous range of scales between η and ℓ_o.
- We will assume that we need n grid points per increment η. Note that n can vary, but a value of 3 to 5 is often suggested.
- Thus, in each direction, the number of required grid points is

$$N_i = \frac{\ell_o}{(\eta/n)} = n\,\frac{\ell_o}{\eta} \sim n\ \mathsf{Re}^{3/4}$$

- Remember that turbulence is 3D, so the total number of grid points needed to accurately estimate the flow is

$$N = \left(n\ \mathsf{Re}^{3/4}\right)^3 = \boxed{n^3\ \mathsf{Re}^{9/4}}$$

Degrees of freedom and numerical simulations

- Let's revisit our example of a typical atmospheric boundary layer flow:

$$U_o \sim 10 \text{ m s}^{-1}, \ \ell_o \sim 10^3 \text{ m}, \ \nu \sim 10^{-5} \text{ m}^2 \text{ s}^{-1}$$

which gives us,

$$\mathsf{Re} = \frac{U_o \ell_o}{\nu} \sim \frac{(10 \text{ m s}^{-1})(10^3 \text{ m})}{10^{-5} \text{ m}^2 \text{ s}^{-1}} \sim 10^9$$

- thus, the number of grid points required to fully resolve this flow (assuming $n = 3$) is

$$N = 9 \times (10^9)^{9/4} \sim 1.6 \times 10^{21}!!!!!!!$$

Note: current capabilities of modern computing allow for grid sizes with $\mathcal{O}(10^{11})$ points.

Degrees of freedom and numerical simulations

- What does a simulation of a typical atmospheric boundary layer flow using a grid with 1.6×10^{21} points buy? (recall $\eta \sim 0.18$ mm)

  $$l_i = \eta * \left(1.6 \times 10^{21}\right)^{1/3} \sim 2 \text{ km}$$

  This means we can simulate a $2 \text{ km} \times 2 \text{ km} \times 2 \text{ km}$ cube. Think how big the atmosphere is and then be depressed.

Degrees of freedom and numerical simulations

- When will we be able to directly simulate all the scales of motion in a turbulent flow?
- A couple of studies used historical data from the literature to build a model that predicts this question (see Voller and Porté-Agel, 2002 and Bou-Zeid, 2014 handouts).
- VP02 derived a model based on Moore's Law

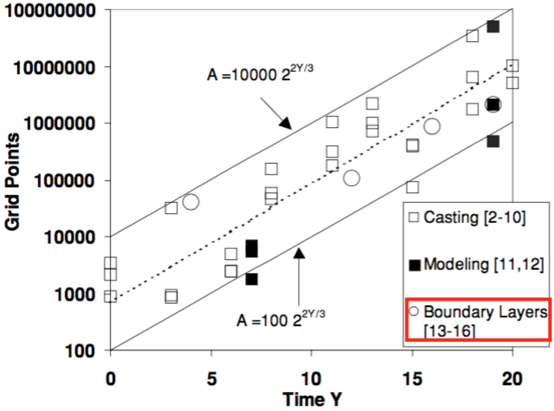
$$P = A\ 2^{0.6667\mathsf{Y}}$$

where A is the computer power at reference year Y=0.

Degrees of freedom and numerical simulations

- VP02 used a reference year of 1980 and used $A = 100$ and $A = 10,000$.
- The best fit was

$$N(t) = 691 \times 2^{0.697(year-1980)}$$



Degrees of freedom and numerical simulations

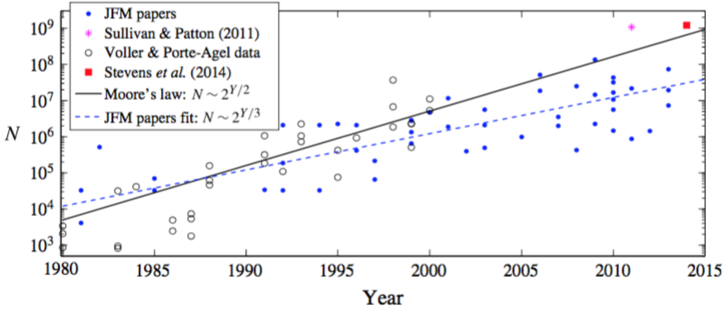
TABLE II

Expected Year (±5) That the Given Direct Simulation Will Be Possible If Grid Size Increases Are Bound by Eq. (2)

Simulation	Domain length scale	Resolution length scale	Grid points required	Expected year (±5 years)
2-D casting	0.1 m	1 μm (dendrite tip)	10^{10}	2015
2-D casting	1 m	1 μm (dendrite tip)	10^{12}	2025
3-D casting	0.1 m	1 μm (dendrite tip)	10^{15}	2040
Boundary layer	100 m	1 mm	10^{15}	2040
2-D casting	0.1 m	1 nm (lattice space)	10^{16}	2045
3-D casting	1 m	1 μm (dendrite tip)	10^{18}	2055
2-D casting	1 m	1 nm (lattice space)	10^{18}	2055
Boundary layer	1 km	1 mm	10^{18}	2055
Boundary layer	10 km	1 mm	10^{21}	2070
3-D casting	0.1 m	1 nm (lattice space)	10^{24}	2085
3-D casting	1 m	1 nm (lattice space)	10^{27}	2100

Degrees of freedom and numerical simulations

- BZ14 updated VP02 using data between 2002 and 2014. It turns out that VP02 was too optimistic. Be more depressed.



Degrees of freedom and numerical simulations

- The Re of relevant flows are orders of magnitude too large for current computational resources.
- Thus, DNS will not be a suitable tool for a long time (relevant to our brief time on Earth).
- The only alternative is to simplify the description of a flow and try to model the small scales instead of resolving them.
- This makes the problem less demanding computationally, but harder in many aspects due to the modeling requirement.
- Before we delve into methods that accomplish this reduction of complexity, we need to understand how we describe a flow.

Equations of motion

- Turbulent flow (and fluid dynamics in general) can be mathematically described by the Navier-Stokes equations (see Bachelor, 1967 for a derivation, see also Pope chapter 2).
- The primary goal of CFD (and LES) is to solve the discretized equations of motion.
- We use the continuum hypothesis (*i.e.*, $\eta \gg$ mean free path of molecules) so that

$$u_i = u_i(x_j, t) \qquad \text{and} \qquad \rho = \rho(x_j, t)$$

Equations of motion: conservation of mass

- Conservation of mass

$$\left.\frac{dm}{dt}\right|_{sys} = 0$$

- We can use Reynolds Transport Theorem (RTT, see any fluids book)

$$\left.\frac{dm}{dt}\right|_{sys} = \frac{\partial}{\partial t} \underbrace{\int_{CV} \rho dV}_{\text{rate of increase in CV}} + \underbrace{\int_{CS} \rho \vec{V} \cdot d\vec{A}}_{\text{net flux leaving CV}} = 0$$

Another way of saying that:
Production + Input = Change (in time) + Output (PICO).

Equations of motion: conservation of mass

- We can use Gauss's theorem and shrink the control volume to an infinitesimal size:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

This is the differential form of the conservation of mass.

Equations of motion: conservation of momentum

- Conservation of momentum (Newton's 2nd law)

$$\sum \vec{F} = \left.\frac{d(m\vec{V})}{dt}\right|_{sys}$$

We can again apply RTT and Gauss's theorem

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i \rho u_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right) - \frac{\partial P}{\partial x_i} + \rho g_i$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is the rate of strain (deformation) tensor.

This is the differential form of the conservation of momentum.

Equations of motion: conservation of energy

- Conservation of energy (1^{st} law of thermodynamics)

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right|_{sys}$$

If we use $e = c_v T$ (specific internal energy and $q_i = -K\partial T/\partial x_i$ (where c_v is the specific heat, T is temperature, and q_i is the thermal flux), then we arrive at

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial}{\partial x_i}\left[u_i(P+E)\right] = \\ \rho\dot{q} + \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}\left[u_j\left(2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right)\right]$$

This is the differential form of the conservation of energy.

Equations of motion: incompressible flow

Let's consider incompressible flow (*i.e.*, the density of a fluid element does not change during its motion)

- Conservation of mass

$$\frac{\partial u_i}{\partial xi} = 0$$

  i.e., divergence of the flow velocity is zero.

- Conservation of momentum

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2} + F_i$$

Equations of motion: incompressible flow

Let's consider incompressible flow (*i.e.*, the density of a fluid element does not change during its motion)

- Conservation of scalar (temperature, species, etc)

$$\frac{\partial \theta}{\partial t} + \frac{\partial u_i \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} + Q$$

where

$$\nu_\theta = \frac{\nu}{\mathsf{Sc}} = \frac{\nu}{\mathsf{Pr}}$$

and Sc is the Schmidt number (used for scalars) and Pr is the Prandtl number (used for temperature).

Equations of motion: incompressible flow

- Recall that

$$\mathsf{Sc} = \frac{\nu}{D} = \frac{\text{viscous diffusion rate}}{\text{molecular diffusion rate}}$$

and

$$\mathsf{Pr} = \frac{\nu}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}$$

Equations of motion: non-dimensional

- We can non-dimensionalize these equations by using a velocity scale (U_o) and length (ℓ_o) scale. For example, the free-stream velocity and the boundary layer depth.
- Conservation of mass

$$\frac{\partial u_i^*}{\partial x_i^*} = 0$$

- Conservation of momentum:

$$\frac{\partial u_i^*}{\partial t*} + \frac{\partial u_i^* u_j^*}{\partial x_j^*} = -\frac{\partial P^*}{\partial x_i^*} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u_i^*}{\partial {x_j^*}^2} + F_i^*$$

where Re is based on our velocity and length scales.

Equations of motion: non-dimensional

- For any general scalar

$$\frac{\partial \theta^*}{\partial t^*} + \frac{\partial u_i^* \theta^*}{\partial x_j^*} = \frac{1}{\text{Sc Re}} \frac{\partial^2 \theta^*}{\partial {x_j^*}^2} + Q^*$$

generally, Sc ~ 1 and Pr ~ 0.72 (for air).

Properties of Navier-Stokes equations

- Reynolds number similarity - for a range of Re, the equations of motion can be considered invariant to transformations of scale.
- Time and space invariance - The equations are invariant to shifts in time or space, *i.e.*, we can define the shifted space variable

$$\hat{x} = \bar{x}/L, \text{where } \bar{x} = x - X$$

- Rotational and reflection invariance - The equations are invariant to rotations and reflections about a fixed axis.
- Invariance to time reflections - The equations are invariant to reflections in time. They are the same going backward or forward in time.
- Galilean invariance - The equations are invariant to constant velocity translations

$$\hat{x} = x - Vt$$

Reynolds number similarity

- As an example of using Reynolds number similarity to make DNS available.
- Recall our example of atmospheric scales that gave a Re of 10^9? We cannot afford this, but if we change the viscosity ν from 10^{-5} to 1, then $\mathrm{Re} = 10^4$ - which is doable.
- In fact, all dimensional scales match those of a typical laboratory experiment.
- We can use Reynolds number similarity to apply findings of our flow using the modified Re to that of a typical atmosphere.

Reynolds number similarity

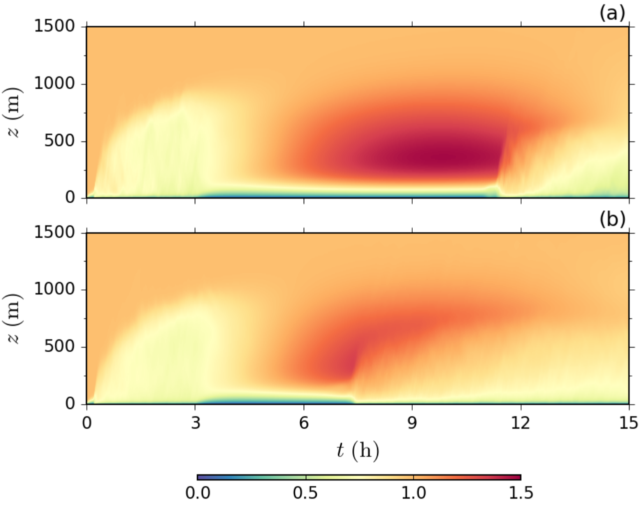


Figure: Velocity from DNS of a low-level jet. (a) and (b) have different slope angles.

Reynolds number similarity

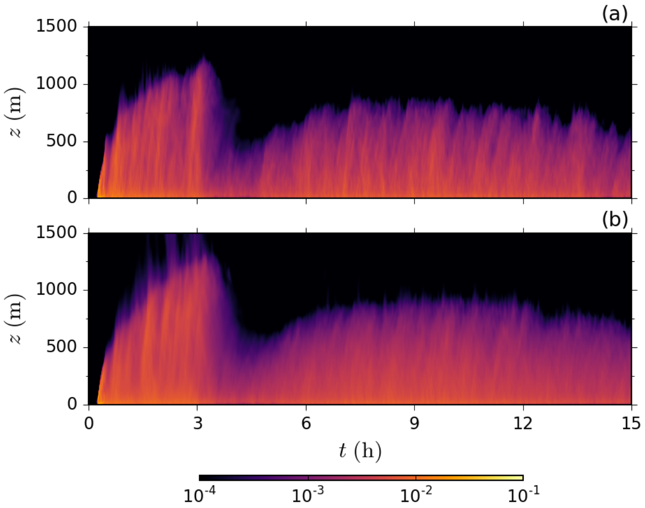


Figure: TKE from DNS of a low-level jet. (a) and (b) have different slope angles.

Reynolds number similarity

- You see that by changing the scaling, we still get results that seem to match the behavior of a flow with a larger Re.
- This is an example of using Reynolds number similarity.