Large-Eddy Simulation of Turbulent Flows

ME EN 7960-003 | Homework #3 Solutions | Due: December 1st

1. Using the basic properties of convolution filters and starting with the scalar conservation equation given by

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = \frac{1}{ScRe} \frac{\partial^2 \theta}{\partial x_i^2} + Q:$$

   (a) derive the filtered scalar conservation equation. Clearly show each step in the process and clearly define the subfilter scale term.

   We apply the filter

$$\widetilde{\frac{\partial \theta}{\partial t}} + \widetilde{u_i \frac{\partial \theta}{\partial x_i}} = \widetilde{\frac{1}{ScRe} \frac{\partial^2 \theta}{\partial x_i^2}} + Q$$

   Note that the filter commutes with differentiation and that:

$$\frac{\partial u_i \theta}{\partial x_i} = u_i \frac{\partial \theta}{\partial x_i} + \theta \cancelto{0}{\frac{\partial u_i}{\partial x_i}} \quad \text{to arrive at} \quad \frac{\partial \widetilde{\theta}}{\partial t} + \frac{\partial \widetilde{u_i \theta}}{\partial x_i} = \frac{1}{ScRe} \frac{\partial^2 \widetilde{\theta}}{\partial x_i^2}$$

   We can decompose the unknown term following Leonard (1974) as

$$\widetilde{u_i \theta} = \widetilde{u}_i \widetilde{\theta} + q_i$$

   where q_i is the SFS scalar flux. The filtered scalar conservation equation is then given by

$$\boxed{\frac{\partial \widetilde{\theta}}{\partial t} + \frac{\partial \widetilde{u}_i \widetilde{\theta}}{\partial x_i} = \frac{1}{ScRe} \frac{\partial^2 \widetilde{\theta}}{\partial x_i^2} - \frac{\partial q_i}{\partial x_i}}$$

   (b) Use Leonard's decomposition (Leonard, Adv. Geophys. 1974) to decompose your answer to (a) and label terms that indicate the SFS Reynolds flux, interaction between resolved and unresolved scales, and the interaction amongst the smallest resolved scales (Leonard term).

   Decompose the parts of the unknown term of q_i by using $u_i = \widetilde{u}_i + u_i'$ and $\theta = \widetilde{\theta} + \theta'$

$$\widetilde{u_i \theta} = \widetilde{(\widetilde{u}_i + u_i')(\widetilde{\theta} + \theta')} = \widetilde{\widetilde{u}_i \widetilde{\theta}} + \widetilde{\widetilde{u}_i \theta'} + \widetilde{u_i' \widetilde{\theta}} + \widetilde{u_i' \theta'}$$

   Using this decomposition, the SFS scalar flux is given by

$$q_i = L_{ij} + C_{ij} + R_{ij}$$

   where

$$L_{ij} = \widetilde{\widetilde{u}_i \widetilde{\theta}} - \widetilde{u}_i \widetilde{\theta} \qquad \rightarrow \text{interaction among the smallest resolved scales}$$

$$C_{ij} = \widetilde{\widetilde{u}_i \theta'} + \widetilde{u_i' \widetilde{\theta}} \qquad \rightarrow \text{interaction among resolved and unresolved scales}$$

$$R_{ij} = \widetilde{u_i' \theta'} \qquad \rightarrow \text{SFS Reynolds flux}$$

2. Using the filtered conservation of momentum equation given by:

$$\frac{\partial \widetilde{u}_i}{\partial t} + \frac{\partial \widetilde{u}_i \widetilde{u}_j}{\partial x_j} = -\frac{\partial \widetilde{P}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \widetilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j} + F_i,$$

derive an equation for the residual kinetic energy $k_r = \frac{1}{2}\tau_{ii}$. Clearly identify the following terms in the equation: all transport terms (i.e., terms that don't create or destroy SFS energy), SFS energy transfer (i.e., Π), and viscous dissipation of energy.

The main idea is to use the relationship $\widetilde{E} = \widetilde{E_f} + k_r$ to derive a balance equation for k_r, where $\widetilde{E}$ is the total filtered KE, $\widetilde{E_f}$ is the resolved TKE, and k_r is the SGS TKE.

First, multiply the filtered momentum equation by u_i and follow the steps in Lecture 7 to arrive at

$$\underbrace{\frac{\partial \widetilde{E}_f}{\partial t}}_{\text{A1}} + \underbrace{\frac{\partial(\widetilde{u}_j \widetilde{E}_f)}{\partial x_j}}_{\text{A2}} = -\underbrace{\frac{\partial(\widetilde{u}_i \widetilde{p})}{\partial x_i}}_{\text{A3}} - \underbrace{\frac{2}{\text{Re}}\frac{\partial(\widetilde{u}_i \widetilde{S}_{ij})}{\partial x_j}}_{\text{A4}} - \underbrace{\epsilon_f}_{\text{A5}} - \underbrace{\Pi}_{\text{A6}} - \underbrace{\frac{\partial(\widetilde{u}_i \tau_{ij})}{\partial x_j}}_{\text{A7}} \qquad (1)$$

Next, filter the product of u_i and the unfiltered momentum equation

$$\widetilde{u_i \frac{\partial u_i}{\partial t}} + \widetilde{u_i \frac{\partial (u_i u_j)}{\partial x_j}} = -\widetilde{u_i \frac{\partial p}{\partial x_j}} + \widetilde{\frac{u_i}{Re}\frac{\partial^2 u_i}{\partial x_j^2}}$$

Following similar procedures as in Lecture 7, we arrive at

$$\underbrace{\frac{\partial \widetilde{E}}{\partial t}}_{\text{B1}} + \underbrace{\frac{\partial(\widetilde{u_j E})}{\partial x_j}}_{\text{B2}} = -\underbrace{\frac{\partial(\widetilde{u_i p})}{\partial x_i}}_{\text{B3}} - \underbrace{\frac{2}{\text{Re}}\frac{\partial(\widetilde{u_i S_{ij}})}{\partial x_j}}_{\text{B4}} - \underbrace{\epsilon}_{\text{B5}} \qquad (2)$$

Finally, subtract Eq. (1) from Eq. (2)

$$\text{B1} - \text{A1}: \frac{\partial \widetilde{E}}{\partial t} - \frac{\partial \widetilde{E}_f}{\partial t} = \frac{\partial(\widetilde{E} - \widetilde{E}_f)}{\partial t}\partial t = \frac{\partial \widetilde{k}_r}{\partial t}$$

$$\text{B2} - \text{A2}: \frac{\partial(\widetilde{u_j E})}{\partial x_j} - \frac{\partial(\widetilde{u}_j \widetilde{E}_f)}{\partial x_j} = \frac{\partial(\widetilde{u_j E})}{\partial x_j} - \frac{\partial \widetilde{u}_j(\widetilde{E} - k_r)}{\partial x_j} = \frac{\partial(\widetilde{u_j E} - \widetilde{u}_j \widetilde{E})}{\partial x_j} + \frac{\partial \widetilde{k}_r}{\partial x_j}$$

$$\text{B3} - \text{A3}: \frac{\partial(\widetilde{u_i p})}{\partial x_i} - \frac{\partial(\widetilde{u}_i \widetilde{p})}{\partial x_i} = \frac{\partial(\widetilde{u_i p} - \widetilde{u}_i \widetilde{p})}{\partial x_i}$$

$$\text{B4} - \text{A4}: \frac{2}{\text{Re}}\frac{\partial(\widetilde{u_i S_{ij}})}{\partial x_j} - \frac{2}{\text{Re}}\frac{\partial(\widetilde{u}_i \widetilde{S}_{ij})}{\partial x_j} = \frac{2}{\text{Re}}\frac{\partial(\widetilde{u_i S_{ij}} - \widetilde{u}_i \widetilde{S}_{ij})}{\partial x_j}$$

$$\text{B5} - \text{A5}: \epsilon - \epsilon_f = \epsilon_r$$

Finally we put everything together

$$\underbrace{\frac{\partial \widetilde{k}_r}{\partial t}}_{\text{C1}} + \underbrace{\frac{\partial \widetilde{k}_r}{\partial x_j}}_{\text{C2}} = \underbrace{\frac{\partial(\widetilde{u_j E} - \widetilde{u}_j \widetilde{E})}{\partial x_j}}_{C3} - \underbrace{\frac{\partial(\widetilde{u_i p} - \widetilde{u}_i \widetilde{p})}{\partial x_i}}_{\text{C4}} - \underbrace{\frac{2}{\text{Re}}\frac{\partial(\widetilde{u_i S_{ij}} - \widetilde{u}_i \widetilde{S}_{ij})}{\partial x_j}}_{\text{C5}} - \underbrace{\frac{\partial(\widetilde{u}_i \tau_{ij})}{\partial x_j}}_{\text{C6}} - \underbrace{\epsilon_r}_{\text{C7}} - \underbrace{\Pi}_{\text{C8}}$$

C1 is storage, C2 is advection, C3 is energy transport, C4 is pressure transport, C5 is viscous stress transport, C6 is SFS stress transport, C7 is dissipation by viscous stress, C8 is SFS dissipation

3. Derive an equation for the SFS scalar flux bulk coefficient $C_s^2 Sc_{\text{sfs}}^{-1}$ that appears in the Smagorinsky eddy-diffusivity model given by:

$$q_i = -\Delta^2 C_s^2 Sc_{\text{sfs}}^{-1} |\widetilde{S}| \frac{\partial \widetilde{\theta}}{\partial x_i},$$

using the dynamic procedure (assume scale invariance). Clearly list any assumptions that you make along the way.

The actual SFS scalar flux is given by

$$q_i = \widetilde{u_i \theta} - \widetilde{u}_i \widetilde{\theta}$$

We can also write the SFS scalar flux at the test filter $(\alpha\Delta)$ as

$$Q_i = \overline{\widetilde{u_i \theta}} - \overline{\widetilde{u}}_i \overline{\widetilde{\theta}}$$

and consider the stress at the smallest resolved scales

$$L_{i\theta} = \overline{\widetilde{u}_i \widetilde{\theta}} - \overline{\widetilde{u}}_i \overline{\widetilde{\theta}}$$

We combine these algebraically to form the Germano identity

$$L_{i\theta} = Q_i - \overline{q}_i$$

First, we assume that the same model can be applied at Δ and $\alpha\Delta$. Using the Smagorinsky model,

$$L_{i\theta} = -\alpha^2 \Delta^2 C_s^2 Sc_{\text{sfs}}^{-1} |\overline{\widetilde{S}}| \frac{\partial \overline{\widetilde{\theta}}}{\partial x_i} + \Delta^2 C_s^2 Sc_{\text{sfs}}^{-1} \overline{|\widetilde{S}| \frac{\partial \widetilde{\theta}}{\partial x_i}}$$

We also have assumed that C_s is applied the same at different filter widths (scale-invariance) and that C_s is constant across the test filter width $\alpha\Delta$ (denoted by $-$). Next, we define the error as

$$e_i = L_{i\theta} - C_s^2 Sc_{\text{sfs}}^{-1} M_i$$

where

$$M_i = \Delta^2 \left[\overline{|\widetilde{S}| \frac{\partial \widetilde{\theta}}{\partial x_i}} - \alpha^2 |\overline{\widetilde{S}}| \frac{\partial \overline{\widetilde{\theta}}}{\partial x_i} \right]$$

Following Lilly (1992), we apply a least-squares approach to minimize the error

$$e_i^2 = L_{i\theta}^2 - 2 C_s^2 Sc_{\text{sfs}}^{-1} L_{i\theta} M_i + (C_s^2 Sc_{\text{sfs}}^{-1})^2 M_i M_i$$

We want the minimum w.r.t. $C_s^2 Sc_{\text{sfs}}^{-1}$:

$$\frac{\partial e_i^2}{\partial (C_s^2 Sc_{\text{sfs}}^{-1})} = -2 L_{i\theta} M_i + 2 C_s^2 Sc_{\text{sfs}}^{-1} M_i M_i = 0$$

Which yields

$$C_s^2 Sc_{\text{sfs}}^{-1} = \frac{L_{i\theta} M_i}{M_i M_i}$$

Since the assumption of constant $C_s Sc_{\text{sfs}}^{-1}$ contributes to numerical instability, we can apply an averaging operator

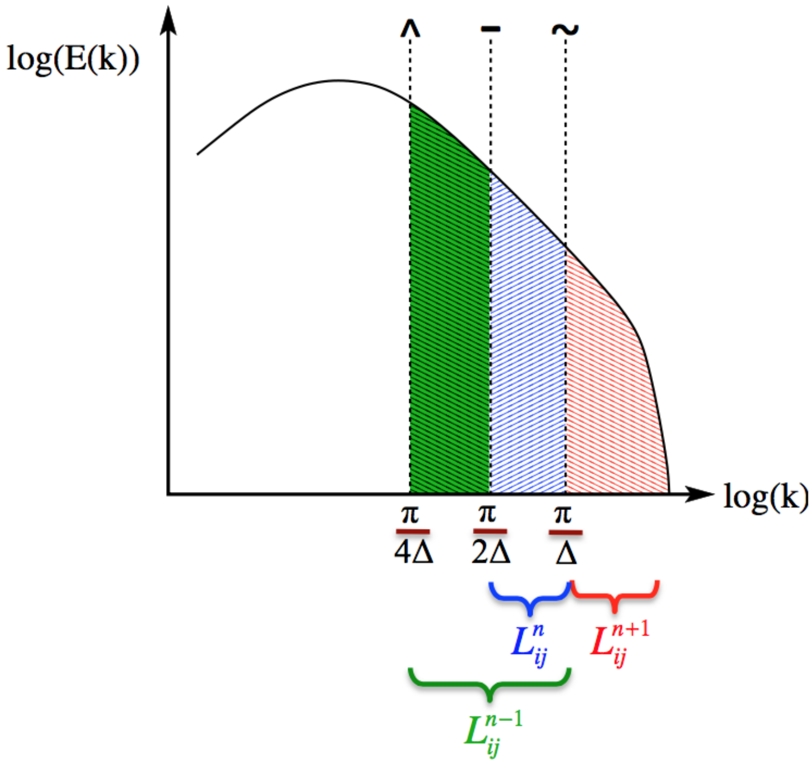
$$C_s^2 Sc_{\text{sfs}}^{-1} = \frac{\langle L_{i\theta} M_i \rangle}{\langle M_i M_i \rangle}$$

4. Starting with SFS scalar flux given by:

$$q_i = \widetilde{u_i\theta} - \widetilde{u}_i\widetilde{\theta},$$

derive a scale similarity model following Liu et al., (J. Fluid Mech. 1994). Clearly state all assumptions.

Consider the following bands around the cutoff filter Δ



Liu et al. (1994) found that the band between Δ ($\sim$) and 4Δ (^) provided the best estimate. For q_i:

$$\begin{aligned} q_i^{n-1} &= \overline{(\widetilde{u}_i - \hat{u}_i)(\widetilde{\theta} - \hat{\theta})} - \overline{(\widetilde{u}_i - \hat{u}_i)}\,\overline{(\widetilde{\theta} - \hat{\theta})} \\ &= \overline{\widetilde{u}_i\widetilde{\theta} - \hat{u}_i\widetilde{\theta} - \widetilde{u}_i\hat{\theta} + \hat{u}_i\hat{\theta}} - \overline{(\widetilde{u}_i - \hat{u}_i)}\,\overline{(\widetilde{\theta} - \hat{\theta})} \end{aligned}$$

Assume (^) terms are constant under the (-) operator

$$\begin{aligned} q_i &= \overline{\widetilde{u}_i\widetilde{\theta}} - \cancel{\hat{u}_i\overline{\widetilde{\theta}}} - \cancel{\overline{\widetilde{u}_i}\hat{\theta}} - \cancel{\hat{u}_i\hat{\theta}} - \overline{\widetilde{u}_i}\,\overline{\widetilde{\theta}} + \cancel{\hat{u}_i\overline{\widetilde{\theta}}} + \cancel{\overline{\widetilde{u}_i}\hat{\theta}} + \cancel{\hat{u}_i\hat{\theta}} \\ q_i &= \overline{\widetilde{u}_i\widetilde{\theta}} - \overline{\widetilde{u}_i}\,\overline{\widetilde{\theta}} \\ q_i &= L_{i\theta} \end{aligned}$$

We assume that there is a similarity between q_i and $L_{i\theta}$. We choose a linear relationship

$$\boxed{q_i = C_L L_{i\theta}}$$

where experimental data has shown $C_L \sim 1$.

5. Derive a scale-dependent dynamic model for the SFS stress based on the Wong-Lilly model (Wong, Lilly; Phys. Fluids, 1994) for the SFS stress given by:

$$\tau_{ij} - \frac{1}{3}\tau_{kk} = -2C_\epsilon \Delta^{4/3}\widetilde{S}_{ij}$$

You will need to use Germano's identity (Germano et al., Phys. Fluids, 1991) at two different scales to do this and you should end up with an algebraic expression for C_ϵ, where C_ϵ is a function of Δ (and the resolved velocity field). Clearly state all assumptions that you make along the way.
Using the Germano identity, $L_{ij} = T_{ij} - \overline{\tau}_{ij}$, where L_{ij} is the Leonard stress and T_{ij} is the SFS stress term applied at a test filter (denoted $-$) width ($\alpha\Delta$).

Let's look at the Germano identity for a filter width of Δ ($\sim$) and a test filter width of 2Δ ($-$). We will assume that the Wong-Lilly model is applicable at our multiple filter widths. Scale-invariance is not assumed, so we introduce a scale-dependent term that follows a power-law distribution at the smallest resolved scales (*i.e.*, $C_{\epsilon,2\Delta}/C_\epsilon = C_{\epsilon,4\Delta}/C_{\epsilon,2\Delta}$), which is given by $\beta = C_{\epsilon,2\Delta}/C_\epsilon$ and $\beta^2 = C_{\epsilon,4\Delta}/C_\epsilon$:

$$\begin{aligned}
T_{ij}(2\Delta) &= -2C_{\epsilon,2\Delta}(2\Delta)^{4/3}\overline{\widetilde{S}}_{ij} \\
\bar{\tau}_{ij} &= -2C_\epsilon \Delta^{4/3}\overline{\widetilde{S}}_{ij} \\
L_{ij,2\Delta} &= T_{ij}(2\Delta) - \bar{\tau}_{ij} \\
&= -2C_{\epsilon,2\Delta}(2\Delta)^{4/3}\overline{\widetilde{S}}_{ij} + 2C_\epsilon\Delta^{4/3}\overline{\widetilde{S}}_{ij} \\
&= -2C_\epsilon 2^{4/3}\Delta^{4/3}\overline{\widetilde{S}}_{ij}\beta + 2C_\epsilon\Delta^{4/3}\overline{\widetilde{S}}_{ij} \\
&\boxed{= -2C_\epsilon\Delta^{4/3}\overline{\widetilde{S}}_{ij}(2^{4/3}\beta - 1)}
\end{aligned}$$

Similarly, when applied at a test filter width of 4Δ:

$$\begin{aligned}
T_{ij}(4\Delta) &= -2C_{\epsilon,4\Delta}(4\Delta)^{4/3}\overline{\widetilde{S}}_{ij} \\
\bar{\tau}_{ij} &= -2C_\epsilon \Delta^{4/3}\overline{\widetilde{S}}_{ij} \\
L_{ij,2\Delta} &= T_{ij}(2\Delta) - \bar{\tau}_{ij} \\
&= -2C_{\epsilon,4\Delta}(4\Delta)^{4/3}\overline{\widetilde{S}}_{ij} + 2C_\epsilon\Delta^{4/3}\overline{\widetilde{S}}_{ij} \\
&= -2C_\epsilon 4^{4/3}\Delta^{4/3}\overline{\widetilde{S}}_{ij}\beta^2 + 2C_\epsilon\Delta^{4/3}\overline{\widetilde{S}}_{ij} \\
&\boxed{= -2C_\epsilon\Delta^{4/3}\overline{\widetilde{S}}_{ij}(4^{4/3}\beta^2 - 1)}
\end{aligned}$$

Next, we define our error for the 2Δ test filter, assume the Leonard stress is trace-free, and minimize the error using a least-squares approach:

$$e_{ij}^2 = (L_{ij} - C_\epsilon M_{ij})^2, \quad \text{where} \quad M_{ij} = -2\Delta^{4/3}\overline{\widetilde{S}}_{ij}(2^{4/3}\beta - 1)$$

$$\frac{\partial e_{ij}^2}{\partial C_\epsilon} = -2L_{ij}M_{ij} + 2C_\epsilon M_{ij}M_{ij} = 0$$

$$C_\epsilon = \frac{\langle L_{ij}M_{ij}\rangle}{\langle M_{ij}M_{ij}\rangle}$$

We repeat the procedure for the 4Δ test filter:

$$e_{ij}^2 = (L_{ij} - C_\epsilon N_{ij})^2, \quad \text{where} \quad N_{ij} = -2\Delta^{4/3}\overline{\widetilde{\overline{S}}}_{ij}(4^{4/3}\beta^2 - 1)$$

$$\frac{\partial e_{ij}^2}{\partial C_\epsilon} = -2L_{ij}N_{ij} + 2C_\epsilon N_{ij}N_{ij} = 0$$

$$C_\epsilon = \frac{\langle L_{ij}N_{ij}\rangle}{\langle N_{ij}N_{ij}\rangle}$$

Equating the two expressions for C_ϵ yields

$$\langle L_{ij}M_{ij}\rangle\langle N_{ij}N_{ij}\rangle - \langle L_{ij}N_{ij}\rangle\langle M_{ij}M_{ij}\rangle = 0$$

From this we can construct an algebraic expression for β via M_{ij} and N_{ij}. Once we have β, we can compute M_{ij}, and thus C_ϵ