# Appendix D
# Fourier transforms

The purpose of this appendix is to provide definitions and a summary of the properties of the Fourier transforms used elsewhere in this book. For further explanations and results, the reader may refer to standard texts, e.g., Bracewell (1965), Lighthill (1970), and Priestley (1981).

*Definition*

Given a function $f(t)$, its Fourier transform is

$$g(\omega) = \mathcal{F}\{f(t)\} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t, \tag{D.1}$$

and the inverse transform is

$$f(t) = \mathcal{F}^{-1}\{g(\omega)\} = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\,\mathrm{d}\omega. \tag{D.2}$$

For $f(t)$ and $g(\omega)$ to form a Fourier-transform pair it is necessary that the above integrals converge, at least as generalized functions. The transforms shown here are between the time domain ($t$) and the frequency domain ($\omega$); the corresponding formulae for transforms between physical space ($x$) and wavenumber space ($\kappa$) are obvious. Some useful Fourier-transform pairs are given in Table D.1 on page 679, and a comprehensive compilation is provided by Erdélyi, Oberhettinger, and Tricomi (1954).

There is not a unique convention for the definition of Fourier transforms. In some definitions the negative exponent $-i\omega t$ appears in the inverse transform, and the factor of $2\pi$ can be split in different ways between $\mathcal{F}$ and $\mathcal{F}^{-1}$. The convention used here is the same as that used by Batchelor (1953), Monin and Yaglom (1975), and Tennekes and Lumley (1972).

Table D.1. *Fourier-transform pairs ($a$, $b$, and $\nu$ are real constants with $b > 0$ and $\nu > -\frac{1}{2}$)*

| $f(t)$ | $g(\omega)$ |
|---|---|
| $1$ | $\delta(\omega)$ |
| $\delta(t-a)$ | $\frac{1}{2\pi}e^{-i\omega a}$ |
| $\delta^{(n)}(t-a)$ | $\frac{(i\omega)^n}{2\pi}e^{-i\omega a}$ |
| $e^{-b\lvert t\rvert}$ | $\frac{b}{\pi(b^2+\omega^2)}$ |
| $\frac{1}{b\sqrt{2\pi}}e^{-t^2/(2b^2)}$ | $\frac{1}{2\pi}e^{-b^2\omega^2/2}$ |
| $H(b-\lvert t\rvert)$ | $\frac{\sin(b\omega)}{\pi\omega}$ |
| $(b^2+t^2)^{-(\nu+1/2)}$ | $\frac{2\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{\lvert\omega\rvert}{2b}\right)^{\nu}K_\nu\left(\frac{\lvert\omega\rvert}{b}\right)$ |

*Derivatives*

The Fourier transforms of derivatives are

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (i\omega)^n g(\omega), \tag{D.3}$$

$$\mathcal{F}^{-1}\left\{\frac{d^n g(\omega)}{d\omega^n}\right\} = (-it)^n f(t). \tag{D.4}$$

*The cosine transform*

If $f(t)$ is real, then $g(\omega)$ has conjugate symmetry:

$$g(\omega) = g^*(-\omega), \quad \text{for } f(t) \text{ real}, \tag{D.5}$$

as may be seen by taking the complex conjugate of Eq. (D.2). If $f(t)$ is real and even (i.e., $f(t) = f(-t)$), then Eq. (D.1) can be rewritten

$$\begin{aligned} g(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt \\ &= \frac{1}{\pi}\int_0^{\infty} f(t)\cos(\omega t)\,dt, \end{aligned} \tag{D.6}$$

showing that $g(\omega)$ is also real and even. The inverse transform is

$$f(t) = 2\int_0^\infty g(\omega)\cos(\omega t)\,\mathrm{d}\omega. \tag{D.7}$$

Equations (D.6) and (D.7) define the *cosine Fourier transform* and its inverse.

In considering spectra, it is sometimes convenient to consider *twice* the Fourier transform of a real even function $f(t)$, i.e.,

$$\begin{aligned}\bar{g}(\omega) &\equiv 2g(\omega)\\ &= \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)\,\mathrm{d}t,\end{aligned} \tag{D.8}$$

so that the inversion formula

$$f(t) = \int_0^\infty \bar{g}(\omega)\cos(\omega t)\,\mathrm{d}\omega, \tag{D.9}$$

does not contain a factor of 2 (cf. Eq. (D.7)).

*The delta function*

The Fourier transform of the delta function $\delta(t-a)$ is (from Eq. (D.1) and invoking the sifting property Eq. (C.11))

$$\mathcal{F}\{\delta(t-a)\} = \frac{1}{2\pi}e^{-i\omega a}, \tag{D.10}$$

and, in particular,

$$\mathcal{F}\{\delta(t)\} = \frac{1}{2\pi}. \tag{D.11}$$

Setting $g(\omega) = (1/2\pi)e^{-i\omega a}$, the inversion formula (Eq. (D.2)) yields

$$\delta(t-a) = \int_{-\infty}^{\infty}\frac{1}{2\pi}e^{i\omega(t-a)}\,\mathrm{d}\omega. \tag{D.12}$$

This is a remarkable and valuable result. However, since the integral in Eq. (D.12) is clearly divergent, it – like $\delta(t-a)$ – must be viewed as a generalized function. That is, with $G(t)$ being a test function, Eq. (D.12) has the meaning

$$\begin{aligned}\int_{-\infty}^{\infty} G(t)\delta(t-a)\,\mathrm{d}t &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}G(t)e^{i\omega(t-a)}\,\mathrm{d}\omega\,\mathrm{d}t\\ &= G(a).\end{aligned} \tag{D.13}$$

Further explanation is provided by Lighthill (1970) and Butkov (1968).

*Convolution*

Given two functions $f_{\mathrm{a}}(t)$ and $f_{\mathrm{b}}(t)$ (both of which have Fourier transforms) their convolution is defined by

$$h(t) \equiv \int_{-\infty}^{\infty} f_{\mathrm{a}}(t-s) f_{\mathrm{b}}(s)\, \mathrm{d}s. \tag{D.14}$$

With the substitution $r = t - s$, the Fourier transform of the convolution is

$$\begin{aligned}
\mathcal{F}\{h(t)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\int_{-\infty}^{\infty} f_{\mathrm{a}}(t-s) f_{\mathrm{b}}(s)\, \mathrm{d}s\, \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega(r+s)} f_{\mathrm{a}}(r) f_{\mathrm{b}}(s)\, \mathrm{d}s\, \mathrm{d}r \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega r} f_{\mathrm{a}}(r)\, \mathrm{d}r \int_{-\infty}^{\infty} e^{-i\omega s} f_{\mathrm{b}}(s)\, \mathrm{d}s \\
&= 2\pi \mathcal{F}\{f_{\mathrm{a}}(t)\}\mathcal{F}\{f_{\mathrm{b}}(t)\}.
\end{aligned} \tag{D.15}$$

That is, the Fourier transform of the convolution is equal to the product of $2\pi$ and the Fourier transforms of the functions.

*Parseval's theorems*

We consider the integral of the product of two functions $f_{\mathrm{a}}(t)$ and $f_{\mathrm{b}}(t)$ that have Fourier transforms $g_{\mathrm{a}}(\omega)$ and $g_{\mathrm{b}}(\omega)$. By writing $f_{\mathrm{a}}$ and $f_{\mathrm{b}}$ as inverse Fourier transforms, we obtain

$$\int_{-\infty}^{\infty} f_{\mathrm{a}}(t) f_{\mathrm{b}}(t)\, \mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_{\mathrm{a}}(\omega) e^{i\omega t}\, \mathrm{d}\omega \int_{-\infty}^{\infty} g_{\mathrm{b}}(\omega') e^{i\omega' t}\, \mathrm{d}\omega'\, \mathrm{d}t \tag{D.16}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_{\mathrm{a}}(\omega) g_{\mathrm{b}}(-\omega'') e^{i(\omega-\omega'')t}\, \mathrm{d}\omega\, \mathrm{d}\omega''\, \mathrm{d}t. \tag{D.17}$$

The integral of the exponential term over all $t$ yields $2\pi\delta(\omega - \omega'')$, see Eq. (D.12), so that the integration of all $\omega''$ is readily performed, producing

$$\int_{-\infty}^{\infty} f_{\mathrm{a}}(t) f_{\mathrm{b}}(t)\, \mathrm{d}t = 2\pi \int_{-\infty}^{\infty} g_{\mathrm{a}}(\omega) g_{\mathrm{b}}(-\omega)\, \mathrm{d}\omega. \tag{D.18}$$

This is *Parseval's second theorem.*

For the case in which $f_{\mathrm{a}}$ and $f_{\mathrm{b}}$ are the same function (i.e., $f_{\mathrm{a}} = f_{\mathrm{b}} = f$ and correspondingly $g_{\mathrm{a}} = g_{\mathrm{b}} = g$), Eq. (D.18) becomes *Parseval's first theorem*:

$$\int_{-\infty}^{\infty} f(t)^2\, \mathrm{d}t = 2\pi \int_{-\infty}^{\infty} g(\omega) g(-\omega)\, \mathrm{d}\omega. \tag{D.19}$$

If $f(t)$ is real, this can be re-expressed as

$$\int_{-\infty}^{\infty} f(t)^2 \, dt = 2\pi \int_{-\infty}^{\infty} g(\omega) g^*(\omega) \, d\omega$$
$$= 4\pi \int_{0}^{\infty} g(\omega) g^*(\omega) \, d\omega. \tag{D.20}$$

**EXERCISES**

D.1 With $f(t)$ being a differentiable function with Fourier transform $g(\omega)$, obtain the following results:

$$f(0) = \int_{-\infty}^{\infty} g(\omega) \, d\omega, \tag{D.21}$$

$$\int_{-\infty}^{\infty} f(t) \, dt = 2\pi g(0), \tag{D.22}$$

$$\int_{-\infty}^{\infty} \left( \frac{d^n f}{dt^n} \right)^2 dt = 2\pi \int_{-\infty}^{\infty} \omega^{2n} g(\omega) g(-\omega) \, d\omega. \tag{D.23}$$

Re-express the right-hand sides for the case of $f(t)$ being real.

D.2 Let $f_a(t)$ be the zero-mean Gaussian distribution with standard deviation $a$, i.e.,

$$f_a(t) = \mathcal{N}(t; 0, a^2) \equiv \frac{1}{a\sqrt{2\pi}} e^{-t^2/(2a^2)}, \tag{D.24}$$

and let $f_b(t) = \mathcal{N}(t; 0, b^2)$, where $a$ and $b$ are positive constants. Show that the convolution of $f_a$ and $f_b$ is $\mathcal{N}(t; 0, a^2 + b^2)$.

# Appendix E
# Spectral representation of stationary random processes

The purpose of this appendix is to show the connections among a statistically-stationary random process $U(t)$, its spectral representation (in terms of Fourier modes), its frequency spectrum $E(\omega)$, and its autocorrelation function $R(s)$.

A statistically stationary random process has a constant variance, and hence does not decay to zero as $|t|$ tends to infinity. As a consequence, the Fourier transform of $U(t)$ does not exist. This fact causes significant technical difficulties, which can be overcome only with more elaborate mathematical tools than are appropriate here. We circumvent this difficulty by first developing the ideas for periodic functions, and then extending the results to the non-periodic functions of interest.

## E.1 Fourier series

We start by considering a non-random real process $U(t)$ in the time interval $0 \le t < T$. The process is continued periodically by defining

$$U(t + NT) = U(t), \tag{E.1}$$

for all non-zero integer $N$.

The time average of $U(t)$ over the period is defined by

$$\langle U(t) \rangle_T \equiv \frac{1}{T} \int_0^T U(t)\, \mathrm{d}t, \tag{E.2}$$

and time averages of other quantities are defined in a similar way. The fluctuation in $U(t)$ is defined by

$$u(t) = U(t) - \langle U(t) \rangle_T, \tag{E.3}$$

and clearly its time average, $\langle u(t) \rangle_T$, is zero.

For each integer $n$, the frequency $\omega_n$ is defined by

$$\omega_n = 2\pi n/T. \tag{E.4}$$

We consider both positive and negative $n$, and observe that

$$\omega_{-n} = -\omega_n. \tag{E.5}$$

The $n$th complex *Fourier mode* is

$$\begin{aligned} e^{i\omega_n t} &= \cos(\omega_n t) + i\sin(\omega_n t) \\ &= \cos(2\pi n t/T) + i\sin(2\pi n t/T). \end{aligned} \tag{E.6}$$

Its time average is

$$\begin{aligned} \langle e^{i\omega_n t}\rangle_T &= 1, \quad \text{for} \quad n = 0, \\ &= 0, \quad \text{for} \quad n \neq 0, \\ &= \delta_{n0}, \end{aligned} \tag{E.7}$$

and hence the modes satisfy the orthogonality condition

$$\langle e^{i\omega_n t} e^{-i\omega_m t}\rangle_T = \langle e^{i(\omega_n - \omega_m)t}\rangle_T = \delta_{nm}. \tag{E.8}$$

The process $u(t)$ can be expressed as a Fourier series,

$$u(t) = \sum_{n=-\infty}^{\infty} (a_n + ib_n) e^{i\omega_n t} = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \tag{E.9}$$

where $\{a_n, b_n\}$ are real, and $\{c_n\}$ are the complex *Fourier coefficients*. Since the time average $\langle u(t)\rangle_T$ is zero, it follows from Eq. (E.7) that $c_0$ is also zero. Expanded in sines and cosines, Eq. (E.9) is

$$\begin{aligned} u(t) = &\sum_{n=1}^{\infty} [(a_n + a_{-n}) + i(b_n + b_{-n})]\cos(\omega_n t) \\ &+ \sum_{n=1}^{\infty} [i(a_n - a_{-n}) - (b_n - b_{-n})]\sin(\omega_n t). \end{aligned} \tag{E.10}$$

Since $u(t)$ is real, $c_n$ satisfies *conjugate symmetry*,

$$c_n = c_{-n}^*, \tag{E.11}$$

(i.e., $a_n = a_{-n}$ and $b_n = -b_{-n}$) so that the imaginary terms on the right-hand side of Eq. (E.10) vanish. Thus the Fourier series (Eq. (E.10)) becomes

$$u(t) = 2\sum_{n=1}^{\infty} [a_n \cos(\omega_n t) - b_n \sin(\omega_n t)], \tag{E.12}$$

which can also be written

$$u(t) = 2\sum_{n=1}^{\infty} |c_n| \cos(\omega_n t + \theta_n), \tag{E.13}$$

where the *amplitude* of the $n$th Fourier mode is

$$|c_n| = (c_n c_n^*)^{1/2} = (a_n^2 + b_n^2)^{1/2}, \tag{E.14}$$

and its *phase* is

$$\theta_n = \tan^{-1}(b_n/a_n). \tag{E.15}$$

An explicit expression for the Fourier coefficients is obtained by multiplying Eq. (E.9) by the $-m$th mode and averaging:

$$\begin{aligned} \langle e^{-i\omega_m t} u(t)\rangle_T &= \left\langle \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} e^{-i\omega_m t} \right\rangle_T \\ &= \sum_{n=-\infty}^{\infty} c_n \delta_{nm} = c_m. \end{aligned} \tag{E.16}$$

It is convenient to introduce the operator $\mathcal{F}_{\omega_n}\{\ \ \}$ defined by

$$\mathcal{F}_{\omega_n}\{u(t)\} \equiv \langle u(t) e^{-i\omega_n t}\rangle_T = \frac{1}{T}\int_0^T u(t) e^{-i\omega_n t}\,\mathrm{d}t, \tag{E.17}$$

so that Eq. (E.16) can be written

$$\mathcal{F}_{\omega_n}\{u(t)\} = c_n. \tag{E.18}$$

Thus, the operator $\mathcal{F}_{\omega_n}\{\ \ \}$ determines the Fourier coefficient of the mode with frequency $\omega_n$.

Equation (E.9) is the spectral representation of $u(t)$, giving $u(t)$ as the sum of discrete Fourier modes $e^{i\omega_n t}$, weighted with Fourier coefficients, $c_n$. With the extension to the non-periodic case in mind, the spectral representation can also be written

$$u(t) = \int_{-\infty}^{\infty} z(\omega) e^{i\omega t}\,\mathrm{d}\omega, \tag{E.19}$$

where

$$z(\omega) \equiv \sum_{n=-\infty}^{\infty} c_n \delta(\omega - \omega_n), \tag{E.20}$$

with $\omega$ being the continuous frequency. The integral in Eq. (E.19) is an inverse Fourier transform (cf. Eq. (D.2)), and hence $z(\omega)$ is identified as the Fourier transform of $u(t)$. (See also Exercise E.1.)

## E.2 Periodic random processes

We now consider $u(t)$ to be a statistically stationary, periodic random process. All the results obtained above are valid for each realization of the process. In particular, the Fourier coefficients $c_n$ are given by Eq. (E.16). However, since $u(t)$ is random, the Fourier coefficients $c_n$ are random variables. We now show that the means $\langle c_n \rangle$ are zero, and that the coefficients corresponding to different frequencies are uncorrelated.

The mean of Eq. (E.9) is

$$\langle u(t) \rangle = \sum_{n=-\infty}^{\infty} \langle c_n \rangle e^{i\omega_n t}. \tag{E.21}$$

Recall that $c_0$ is zero, and that for $n \neq 0$, the stationarity condition – that $\langle u(t) \rangle$ be independent of $t$ – evidently implies that $\langle c_n \rangle$ is zero.

The covariance of the Fourier modes is

$$\begin{aligned}
\langle c_n c_m \rangle &= \langle \langle e^{-i\omega_n t} u(t) \rangle_T \langle e^{-i\omega_m t} u(t) \rangle_T \rangle \\
&= \frac{1}{T^2} \int_0^T \int_0^T e^{-i\omega_n t} e^{-i\omega_m t'} \langle u(t) u(t') \rangle \, \mathrm{d}t' \, \mathrm{d}t \\
&= \frac{1}{T} \int_0^T e^{-i(\omega_n + \omega_m)t} \left( \frac{1}{T} \int_{-t}^{T-t} e^{-i\omega_m s} R(s) \, \mathrm{d}s \right) \mathrm{d}t \\
&= \delta_{n(-m)} \mathcal{F}_{\omega_m}\{R(s)\}.
\end{aligned} \tag{E.22}$$

The third line follows from the substitution $t' = t + s$, and from the definition of the autocovariance

$$R(s) \equiv \langle u(t) u(t+s) \rangle, \tag{E.23}$$

which (because of stationarity) is independent of $t$. The integrand $e^{-i\omega_m s} R(s)$ is periodic in $s$, with period $T$, so the integral in large parentheses is independent of $t$. The last line then follows from Eqs. (E.8) and (E.17).

It is immediately evident from Eq. (E.22) that the covariance $\langle c_n c_m \rangle$ is zero unless $m$ equals $-n$: that is, *Fourier coefficients corresponding to different frequencies are uncorrelated*. For $m = -n$, Eq. (E.22) becomes

$$\langle c_n c_{-n} \rangle = \langle c_n c_n^* \rangle = \langle |c_n|^2 \rangle = \mathcal{F}_{\omega_n}\{R(s)\}. \tag{E.24}$$

Thus the variances $\langle |c_n|^2 \rangle$ are the Fourier coefficients of $R(s)$, which can therefore be expressed as

$$R(s) = \sum_{n=-\infty}^{\infty} \langle c_n c_n^* \rangle e^{i\omega_n s} = 2 \sum_{n=1}^{\infty} \langle |c_n|^2 \rangle \cos(\omega_n s). \tag{E.25}$$

It may be observed that $R(s)$ is real and an even function of $s$, and that it depends only on the amplitudes $|c_n|$ independent of the phases $\theta_n$.

Again with the extension to the non-periodic case in mind, we define the *frequency spectrum* by

$$\check{E}(\omega) = \sum_{n=-\infty}^{\infty} \langle c_n c_n^* \rangle \delta(\omega - \omega_n), \tag{E.26}$$

so that the autocovariance can be written

$$R(s) = \sum_{n=-\infty}^{\infty} \langle c_n c_n^* \rangle e^{i\omega_n s} = \int_{-\infty}^{\infty} \check{E}(\omega) e^{i\omega s}\, \mathrm{d}\omega \tag{E.27}$$

(cf. Eq. E.19).

It may be seen from its definition that $\check{E}(\omega)$ is a real, even function of $\omega$ (i.e., $\check{E}(\omega) = \check{E}(-\omega)$). It is convenient, then, to define the (alternative) frequency spectrum by

$$E(\omega) = 2\check{E}(\omega), \quad \text{for } \omega \geq 0, \tag{E.28}$$

and to rewrite Eq. (E.27) as the inverse cosine transform (Eq. (D.9))

$$R(s) = \int_0^{\infty} E(\omega) \cos(\omega s)\, \mathrm{d}\omega. \tag{E.29}$$

Setting $s = 0$ in the above equation, we obtain

$$\begin{aligned} R(0) = \langle u(t)^2 \rangle = \sum_{n=-\infty}^{\infty} \langle c_n c_n^* \rangle &= \int_{-\infty}^{\infty} \check{E}(\omega)\, \mathrm{d}\omega \\ &= \int_0^{\infty} E(\omega)\, \mathrm{d}\omega. \end{aligned} \tag{E.30}$$

Consequently, $\langle c_n c_n^* \rangle$ represents the contribution to the variance from the $n$th mode, and similarly

$$\int_{\omega_a}^{\omega_b} E(\omega)\, \mathrm{d}\omega$$

is the contribution to $\langle u(t)^2 \rangle$ from the frequency range $\omega_a \leq |\omega| < \omega_b$. It is clear from Eq. (E.26) that, like $R(s)$, the spectrum $E(\omega)$ is independent of the phases.

It may be observed that Eq. (E.27) identifies $R(s)$ as the inverse Fourier transform of $\check{E}(\omega)$ (cf. Eq. (D.2)). Hence, as may be verified directly from Eq. (E.25), $\check{E}(\omega)$ is the Fourier transform of $R(s)$:

$$\check{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(s) e^{-i\omega s}\, \mathrm{d}s. \tag{E.31}$$

Similarly, $E(\omega)$ is *twice* the Fourier transform of $R(s)$:

$$E(\omega) = \frac{2}{\pi} \int_0^\infty R(s) \cos(\omega s) \, \mathrm{d}s. \tag{E.32}$$

Having identified $R(s)$ and $\check{E}(\omega)$ as a Fourier-transform pair, we now take Eq. (E.31) as the definition of $\check{E}(\omega)$ (rather than Eq. (E.26)).

The spectrum $\check{E}(\omega)$ can also be expressed in terms of the Fourier transform $z(\omega)$, defined in Eq. (E.20). Consider the infinitesimal interval $(\omega, \ \omega + \mathrm{d}\omega)$, which contains either zero or one of the discrete frequencies $\omega_n$. If it contains none of the discrete frequencies, then

$$z(\omega) \, \mathrm{d}\omega = 0, \quad \check{E}(\omega) \, \mathrm{d}\omega = 0. \tag{E.33}$$

On the other hand, if it contains the discrete frequency $\omega_n$, then

$$z(\omega) \, \mathrm{d}\omega = c_n, \quad \check{E}(\omega) \, \mathrm{d}\omega = \langle c_n c_n^* \rangle. \tag{E.34}$$

Thus, in general,

$$\check{E}(\omega) \, \mathrm{d}\omega = \langle z(\omega) z(\omega)^* \rangle \, \mathrm{d}\omega^2. \tag{E.35}$$

The essential properties of the spectrum are that

(i) $\check{E}(\omega)$ is non-negative ($\check{E}(\omega) \geq 0$) (see Eq. (E.35));
(ii) $\check{E}(\omega)$ is real (because $R(s)$ is even, i.e., $R(s) = R(-s)$); and
(iii) $\check{E}(\omega)$ is even, i.e., $\check{E}(\omega) = \check{E}(-\omega)$ (because $R(s)$ is real).

Table E.1 provides a summary of the relationships among $u(t)$, $c_n$, $z(\omega)$, $R(s)$, and $\check{E}(\omega)$.

### EXERCISES

E.1 By taking the Fourier transform of Eq. (E.9), show that $z(\omega)$ given by Eq. (E.20) is the Fourier transform of $u(t)$. (Hint: see Eq. (D.12).)

E.2 Show that the Fourier coefficients $c_n = a_n + ib_n$ of a statistically stationary, periodic random process satisfy

$$\langle c_n^2 \rangle = 0, \quad \langle a_n^2 \rangle = \langle b_n^2 \rangle, \quad \langle a_n b_n \rangle = 0, \tag{E.36}$$

$$\langle a_n a_m \rangle = \langle a_n b_m \rangle = \langle b_n b_m \rangle = 0, \quad \text{for } n \neq m. \tag{E.37}$$

Table E.1. *Spectral properties of periodic and non-periodic statistically stationary random processes*

| | Periodic | Non-periodic |
|---|---|---|
| Autocovariance | $R(s) \equiv \langle u(t)u(t+s)\rangle$, periodic | $R(s) \equiv \langle u(t)u(t+s)\rangle$, $R(\pm\infty) = 0$ |
| Spectrum | $\check{E}(\omega) \equiv \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} R(s)e^{-i\omega s}\,\mathrm{d}s$, discrete | $\check{E}(\omega) \equiv \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} R(s)e^{-i\omega s}\,\mathrm{d}s$, continuous |
| | $E(\omega) = 2\check{E}(\omega) = \dfrac{2}{\pi}\displaystyle\int_{0}^{\infty} R(s)\cos(\omega s)\,\mathrm{d}s$ | $E(\omega) = 2\check{E}(\omega) = \dfrac{2}{\pi}\displaystyle\int_{0}^{\infty} R(s)\cos(\omega s)\,\mathrm{d}s$ |
| | $R(s) = \displaystyle\int_{-\infty}^{\infty} \check{E}(\omega)e^{i\omega s}\,\mathrm{d}\omega = \int_{0}^{\infty} E(\omega)\cos(\omega s)\,\mathrm{d}\omega$ | $R(s) = \displaystyle\int_{-\infty}^{\infty} \check{E}(\omega)e^{i\omega s}\,\mathrm{d}\omega = \int_{0}^{\infty} E(\omega)\cos(\omega s)\,\mathrm{d}\omega$ |
| Fourier coefficient | $c_n = \langle e^{-i\omega_n t}u(t)\rangle_T = \mathcal{F}_{\omega_n}\{u(t)\}$ | |
| Fourier transform | $z(\omega) = \displaystyle\sum_{n=-\infty}^{\infty} c_n\delta(\omega - \omega_n)$ | |
| Spectral representation | $u(t) = \displaystyle\int_{-\infty}^{\infty} e^{i\omega t}z(\omega)\,\mathrm{d}\omega = \sum_{n=-\infty}^{\infty} e^{i\omega_n t}c_n$ | $u(t) = \displaystyle\int_{-\infty}^{\infty} e^{i\omega t}\,\mathrm{d}Z(\omega)$ |
| Spectrum | $\check{E}(\omega)\,\mathrm{d}\omega = \langle z(\omega)z(\omega)^*\rangle\,\mathrm{d}\omega^2$ | $\check{E}(\omega)\,\mathrm{d}\omega = \langle\,\mathrm{d}Z(\omega)\,\mathrm{d}Z(\omega)^*\rangle$ |

## E.3 Non-periodic random processes

We now consider $u(t)$ to be a non-periodic, statistically stationary random process. Instead of being periodic, the autocovariance $R(s)$ decays to zero as $|s|$ tends to infinity. Just as in the periodic case, the spectrum $\check{E}(\omega)$ is defined to be the Fourier transform of $R(s)$, but now $\check{E}(\omega)$ is a continuous function of $\omega$, rather than being composed of delta functions.

In an approximate sense, the non-periodic case can be viewed as the periodic case in the limit as the period $T$ tends to infinity. The difference between adjacent discrete frequencies is

$$\Delta\omega \equiv \omega_{n+1} - \omega_n = 2\pi/T, \tag{E.38}$$

which tends to zero as $T$ tends to infinity. Consequently, within any given frequency range ($\omega_a \le \omega < \omega_b$), the number of discrete frequencies ($\approx (\omega_b - \omega_a)/\Delta\omega$) tends to infinity, so that (in an approximate sense) the spectrum becomes continuous in $\omega$.

Mathematically rigorous treatments of the non-periodic case are given by Monin and Yaglom (1975) and Priestley (1981). Briefly, while the non-periodic process $u(t)$ does not have a Fourier transform, it does have a spectral representation in terms of the Fourier–Stieltjes integral

$$u(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, \mathrm{d}Z(\omega), \tag{E.39}$$

where $Z(\omega)$ is a non-differentiable complex random function. It may be observed that $\mathrm{d}Z(\omega)$ (in the non-periodic case) corresponds to $z(\omega)\,\mathrm{d}\omega$ (in the periodic case, Eq. (E.20)). The spectrum for the non-periodic case corresponding to Eq. (E.35) for the periodic case is

$$\check{E}(\omega)\,\mathrm{d}\omega = \langle\, \mathrm{d}Z(\omega)\,\mathrm{d}Z(\omega)^* \rangle. \tag{E.40}$$

In Table E.1 the spectral representations for the periodic and non-periodic cases are compared.

## E.4 Derivatives of the process

For the periodic case, the process $u(t)$ has the spectral representation Eq. (E.9). On differentiating with respect to time, we obtain the spectral representation of $\mathrm{d}u/\mathrm{d}t$:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \sum_{n=-\infty}^{\infty} i\omega_n c_n e^{i\omega_n t}. \tag{E.41}$$

Similarly, the spectral representation of the $k$th derivative is

$$u^{(k)}(t) \equiv \frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k} = \sum_{n=-\infty}^{\infty} (i\omega_n)^k c_n e^{i\omega_n t}. \tag{E.42}$$

Because $u^{(k)}(t)$ is determined by the Fourier coefficients of $u(t)$ (i.e., $c_n$ in Eq. (E.42)), the autocovariance and spectrum of $u^{(k)}(t)$ are determined by $R(s)$ and $E(\omega)$.

It follows from the same procedure as that which leads to Eq. (E.25) that the autocorrelation of $u^{(k)}(t)$ is

$$\begin{aligned} R_k(s) &\equiv \langle u^{(k)}(t)u^{(k)}(t+s)\rangle \\ &= \sum_{n=-\infty}^{\infty} \omega_n^{2k}\langle c_n c_n^*\rangle e^{i\omega_n s}. \end{aligned} \tag{E.43}$$

By comparing this with the $(2k)$th derivative of Eq. (E.25),

$$\frac{\mathrm{d}^{2k}R(s)}{\mathrm{d}s^{2k}} = (-1)^k \sum_{n=-\infty}^{\infty} \omega_n^{2k}\langle c_n c_n^*\rangle e^{i\omega_n s}, \tag{E.44}$$

we obtain the result

$$R_k(s) = (-1)^k \frac{\mathrm{d}^{2k}R(s)}{\mathrm{d}s^{2k}}. \tag{E.45}$$

The spectrum of $u^{(k)}(t)$ is (cf. Eq. (E.26))

$$\begin{aligned} \check{E}_k(\omega) &\equiv \sum_{n=-\infty}^{\infty} \omega_n^{2k}\langle c_n c_n^*\rangle \delta(\omega-\omega_n), \\ &= \omega^{2k} \sum_{n=-\infty}^{\infty} \langle c_n c_n^*\rangle \delta(\omega-\omega_n) \\ &= \omega^{2k}\check{E}(\omega), \end{aligned} \tag{E.46}$$

a result that can, alternatively, be obtained by taking the Fourier transform of Eq. (E.45).

In summary, for the $k$th derivative $u^{(k)}(t)$ of the process $u(t)$, the autocovariance $R_k(s)$ is given by Eq. (E.45), while the spectrum is

$$\check{E}_k(\omega) = \omega^{2k}\check{E}(\omega). \tag{E.47}$$

These two results apply both to the periodic and to non-periodic cases.

Appendix F

# The discrete Fourier transform

We consider a periodic function $u(t)$, with period $T$, sampled at $N$ equally spaced times within the period, where $N$ is an even integer. On the basis of these samples, the *discrete Fourier transform* defines $N$ Fourier coefficients (related to the coefficients of the Fourier series) and thus provides a discrete spectral representation of $u(t)$.

The *fast Fourier transform* (FFT) is an efficient implementation of the discrete Fourier transform. In numerical methods, the FFT and its inverse can be used to transform between time and frequency domains, and between physical space and wavenumber space. In DNS and LES of flows with one or more directions of statistical homogeneity, pseudo-spectral methods are generally used (in the homogeneous directions), with FFTs being used extensively to transform between physical and wavenumber spaces.

The time interval $\Delta t$ is defined by

$$\Delta t \equiv \frac{T}{N}, \tag{F.1}$$

the sampling times are

$$t_j \equiv j\Delta t, \quad \text{for } \ j = 0, 1, \ldots, N-1, \tag{F.2}$$

and the samples are denoted by

$$u_j \equiv u(t_j). \tag{F.3}$$

The complex coefficients $\tilde{c}_k$ of the discrete Fourier transform are then defined for $1 - \frac{1}{2}N \leq k \leq \frac{1}{2}N$ by

$$\tilde{c}_k \equiv \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-i\omega_k t_j} = \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-2\pi i jk/N}, \tag{F.4}$$

where (as with Fourier series) the frequency $\omega_k$ is defined by

$$\omega_k = \frac{2\pi k}{T}. \tag{F.5}$$

As demonstrated below, the inverse transform is

$$u_\ell = \sum_{k=1-\frac{1}{2}N}^{\frac{1}{2}N} \tilde{c}_k e^{i\omega_k t_\ell} = \sum_{k=1-\frac{1}{2}N}^{\frac{1}{2}N} \tilde{c}_k e^{2\pi i k \ell/N}. \tag{F.6}$$

In order to confirm the form of the inverse transform, we consider the quantity

$$\mathcal{I}_{j,N} \equiv \frac{1}{N} \sum_{k=1-\frac{1}{2}N}^{\frac{1}{2}N} e^{2\pi i jk/N}. \tag{F.7}$$

Viewed in the complex plane, $\mathcal{I}_{j,N}$ is the centroid of the $N$ points $e^{2\pi i jk/N}$ for the $N$ values of $k$. For $j$ being zero or an integer multiple of $N$, each point is located at $(1, 0)$, so $\mathcal{I}_{j,N}$ is unity. For $j$ not being an integer multiple of $N$, the points are distributed symmetrically about the origin, so $\mathcal{I}_{j,N}$ is zero. Thus

$$\mathcal{I}_{j,N} = \begin{cases} 1, & \text{for } j/N \text{ integer,} \\ 0, & \text{otherwise.} \end{cases} \tag{F.8}$$

With this result, the right-hand side of Eq. (F.6) can be written

$$\begin{aligned} \sum_{k=1-\frac{1}{2}N}^{\frac{1}{2}N} \tilde{c}_k e^{2\pi i k\ell/N} &= \sum_{k=1-\frac{1}{2}N}^{\frac{1}{2}N} \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-2\pi i jk/N} e^{2\pi i k\ell/N} \\ &= \sum_{j=0}^{N-1} u_j \frac{1}{N} \sum_{k=1-\frac{1}{2}N}^{\frac{1}{2}N} e^{2\pi i k(\ell-j)/N} \\ &= \sum_{j=0}^{N-1} u_j \mathcal{I}_{(\ell-j),N} = u_\ell. \end{aligned} \tag{F.9}$$

In the final sum, the only non-zero contribution is for $j = \ell$. This verifies the inverse transform, Eq. (F.6).

It is informative to study the relationship between the coefficients of the discrete Fourier transform $\tilde{c}_k$ and those of the Fourier series $c_k$. From the definitions of these quantities (Eqs. (F.6) and (E.9)) we have

$$u_\ell = \sum_{k=1-\frac{1}{2}N}^{\frac{1}{2}N} \tilde{c}_k e^{i\omega_k t_\ell} = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k t_\ell}. \tag{F.10}$$

Before considering the general case, we consider the simpler situation in which the Fourier coefficients $c_k$ are zero for all modes with $|\omega_k| \geq \omega_{\max}$, where $\omega_{\max}$ is the highest frequency represented in the discrete Fourier transform,

$$\omega_{\max} \equiv \frac{\pi}{\Delta t} = \omega_{N/2}. \tag{F.11}$$

In this case, the sums in Eq. (F.10) are both effectively from $-(\frac{1}{2}N - 1)$ to $(\frac{1}{2}N - 1)$, so the coefficients $\tilde{c}_k$ and $c_k$ are identical.

For the general case, we need to consider frequencies higher than $\omega_{\max}$. For $k$ in the range $-(\frac{1}{2}N - 1) \leq k \leq \frac{1}{2}N$, and for a non-zero integer $m$, the $(k + mN)$th mode has the frequency

$$\omega_{k+mN} = \omega_k + 2m\omega_{\max}, \tag{F.12}$$

with

$$|\omega_{k+mN}| \geq \omega_{\max}. \tag{F.13}$$

At the sampling times $t_j$, the $(k + mN)$th mode is indistinguishable from the $k$th mode, since

$$e^{i\omega_{k+mN} t_j} = e^{2\pi i j(k+mN)/N} = e^{2\pi i jk/N} = e^{i\omega_k t_j}. \tag{F.14}$$

The $(k + mN)$th mode is said to be *aliased* to the $k$th mode.

The coefficients $\tilde{c}_k$ can be determined from their definition (Eq. (F.4)) with the Fourier series substituted for $u_j$:

$$\begin{aligned} \tilde{c}_k &= \frac{1}{N} \sum_{j=0}^{N-1} \left( \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n j} \right) e^{-2\pi i jk/N} \\ &= \sum_{n=-\infty}^{\infty} \mathcal{I}_{n-k,N} c_n = \sum_{m=-\infty}^{\infty} c_{k+mN}. \end{aligned} \tag{F.15}$$

Thus the coefficient $\tilde{c}_k$ is the sum of the Fourier coefficients of all the modes that are aliased to the $k$th mode.

In view of conjugate symmetry, the $N$ complex coefficient $\tilde{c}_k$ can be expressed in terms of $N$ real numbers (e.g., $\Re\{\tilde{c}_k\}$ for $k = 0, 1, \ldots, \frac{1}{2}N$ and $\Im\{\tilde{c}_k\}$ for $k = 1, 2, \ldots, \frac{1}{2}N-1$, see Exercise F.1). The discrete Fourier transform and its inverse provide a one-to-one mapping between $u_j$ and $\tilde{c}_k$. On the order of $N^2$ operations are required in order to evaluate $\tilde{c}_k$ directly from the sum in Eq. (F.4). However, the same result can be obtained in on the order of $N \log N$ operations by using the *fast Fourier transform* (FFT) (see, e.g., Brigham (1974)). Thus, for periodic data sampled at sufficiently small time intervals, the FFT is an efficient tool for evaluating Fourier coefficients, spectra, autocorrelations (as inverse Fourier transforms of spectra), convolutions, and derivatives as

$$\frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n} = \mathcal{F}^{-1}\{(i\omega)^n \mathcal{F}\{u(t)\}\}. \tag{F.16}$$

As with the Fourier transform, there are various definitions of the discrete Fourier transform. The definition used here makes the most direct connection with Fourier series. In numerical implementations, the alternative definition given in Exercise F.2 is usually used.

**EXERCISES**

F.1 Show that, for real $u(t)$, the coefficients $\tilde{c}_k$ satisfy

$$\tilde{c}_k = \tilde{c}^*_{-k}, \quad \text{for } |k| < \tfrac{1}{2}N, \tag{F.17}$$

and that $\tilde{c}_0$ and $\tilde{c}_{\frac{1}{2}N}$ are real.

Show that

$$\cos(\omega_{\max} t_j) = (-1)^j, \tag{F.18}$$

$$\sin(\omega_{\max} t_j) = 0. \tag{F.19}$$

F.2 An alternative definition of the discrete Fourier transform is

$$\overline{c}_k = \sum_{j=0}^{N-1} u_j e^{-2\pi i jk/N}, \quad \text{for } k = 0, 1, \ldots, N-1. \tag{F.20}$$

Show that the inverse is

$$u_\ell = \frac{1}{N} \sum_{k=0}^{N-1} \overline{c}_k e^{2\pi i k\ell/N}. \tag{F.21}$$

What is the relationship between the coefficients $\tilde{c}_k$ and $\overline{c}_k$?

# Appendix G
# Power-law spectra

In the study of turbulence, considerable attention is paid to the shape of spectra at high frequency (or large wavenumber). The purpose of this appendix is to show the relationships among a power-law spectrum ($E(\omega) \sim \omega^{-p}$, for large $\omega$), the underlying random process $u(t)$, and the second-order structure function $D(s)$.

We consider a statistically stationary process $u(t)$ with finite variance $\langle u^2 \rangle$ and integral timescale $\bar{\tau}$. The autocorrelation function

$$R(s) \equiv \langle u(t)u(t+s) \rangle \tag{G.1}$$

and half the frequency spectrum form a Fourier-cosine-transform pair:

$$R(s) = \int_0^\infty E(\omega)\cos(\omega s)\,\mathrm{d}\omega, \tag{G.2}$$

$$E(\omega) = \frac{2}{\pi}\int_0^\infty R(s)\cos(\omega s)\,\mathrm{d}s. \tag{G.3}$$

The third quantity of interest is the second-order structure function

$$\begin{aligned} D(s) &\equiv \langle [u(t+s) - u(t)]^2 \rangle \\ &= 2[R(0) - R(s)] \\ &= 2\int_0^\infty [1 - \cos(\omega s)]E(\omega)\,\mathrm{d}\omega. \end{aligned} \tag{G.4}$$

By definition, a power-law spectrum varies as

$$E(\omega) \sim \omega^{-p}, \quad \text{for large } \omega, \tag{G.5}$$

whereas a power-law structure function varies as

$$D(s) \sim s^q, \quad \text{for small } s. \tag{G.6}$$

The aim here is to understand the significances of particular values of $p$ and $q$ and the connection between them.

The first observation – obtained by setting $s = 0$ in Eq. (G.2) – is that the variance is

$$\langle u^2 \rangle = R(0) = \int_0^\infty E(\omega)\,\mathrm{d}\omega. \tag{G.7}$$

By assumption $\langle u^2 \rangle$ is finite. Hence, if $E(\omega)$ is a power-law spectrum, the requirement that the integral converges dictates $p > 1$.

A sequence of similar results stems from the spectra of the derivatives of $u(t)$. Suppose that the $n$th derivative of $u(t)$ exists, and denote it by

$$u^{(n)}(t) = \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n}. \tag{G.8}$$

The autocorrelation of $u^{(n)}(t)$ is

$$\begin{aligned} R_n(s) &\equiv \langle u^{(n)}(t) u^{(n)}(t+s) \rangle \\ &= (-1)^n \frac{\mathrm{d}^{2n} R(s)}{\mathrm{d}s^{2n}} \end{aligned} \tag{G.9}$$

(see Appendix E, Eq. (E.45)), and its frequency spectrum is

$$E_n(\omega) = \omega^{2n} E(\omega) \tag{G.10}$$

(see Eq. (E.47)). Hence we obtain

$$\begin{aligned} \left\langle \left( \frac{\mathrm{d}^n u}{\mathrm{d}t^n} \right)^2 \right\rangle = R_n(0) &= \int_0^\infty E_n(\omega)\,\mathrm{d}\omega \\ &= \int_0^\infty \omega^{2n} E(\omega)\,\mathrm{d}\omega. \end{aligned} \tag{G.11}$$

The left-hand side is finite if $u(t)$ is differentiable $n$ times (in a mean-square sense). Then, if $E(\omega)$ is a power-law spectrum, the requirement that the integral in Eq. (G.11) converges dictates

$$p > 2n + 1. \tag{G.12}$$

For an infinitely differentiable process – such as the velocity evolving by the Navier–Stokes equations – it follows from Eq. (G.12) that (for large $\omega$) the spectrum decays more rapidly than any power of $\omega$: it may instead decay as $\exp(-\omega)$ or $\exp(-\omega^2)$, for example. Nevertheless, over a significant range of frequencies ($\omega_\mathrm{l} < \omega < \omega_\mathrm{h}$, say) a power-law spectrum may occur, with exponential decay beyond $\omega_\mathrm{h}$.

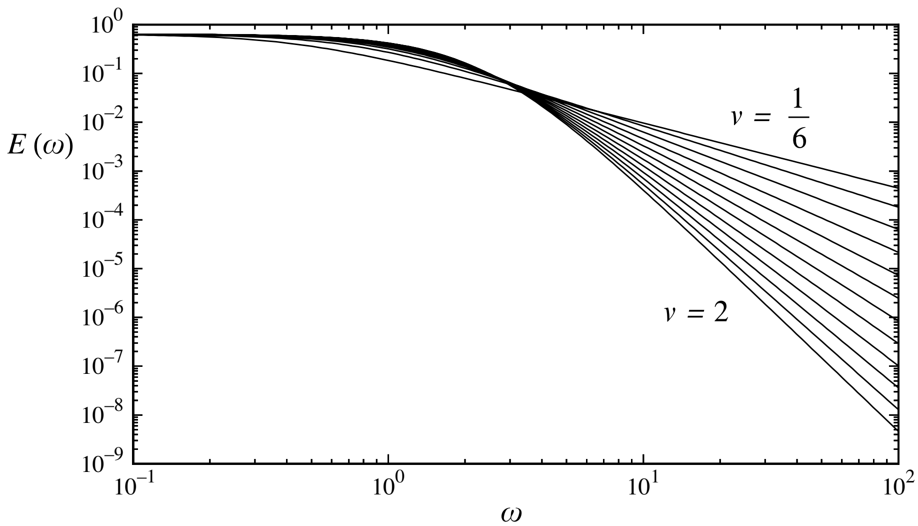


Fig. G.1. Non-dimensional power-law spectra $E(\omega)$: Eq. (G.14) for $\nu = \frac{1}{6}, \frac{1}{3}, \ldots, 1\frac{5}{6}, 2$.

If the process $u(t)$ is at least once continuously differentiable, then, for small $s$, the structure function is

$$D(s) \approx \left\langle \left( \frac{\mathrm{d}u}{\mathrm{d}t} \right)^2 \right\rangle s^2, \tag{G.13}$$

i.e., a power law (Eq. (G.6)) with $q = 2$.

It is instructive to examine the non-dimensional power-law spectrum

$$E(\omega) = \frac{2}{\pi} \left( \frac{\alpha^2}{\alpha^2 + \omega^2} \right)^{(1+2\nu)/2}, \tag{G.14}$$

for various values of the positive parameter $\nu$. The non-dimensional integral timescale is unity, i.e.,

$$\int_0^\infty R(s)\,\mathrm{d}s = \frac{\pi}{2} E(0) = 1, \tag{G.15}$$

while (for given $\nu$) $\alpha$ is specified (see Eq. (G.18) below) so that the variance is unity. Figure G.1 shows $E(\omega)$ for $\nu$ between $\frac{1}{6}$ and 2. For large $\omega$, the straight lines on the log–log plot clearly show the power-law behavior with

$$p = 1 + 2\nu. \tag{G.16}$$

The corresponding autocorrelation (obtained as the inverse transform of Eq. (G.14)) is

$$R(s) = \frac{2\alpha}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \left(\tfrac{1}{2} s\alpha\right)^\nu K_\nu(s\alpha), \tag{G.17}$$

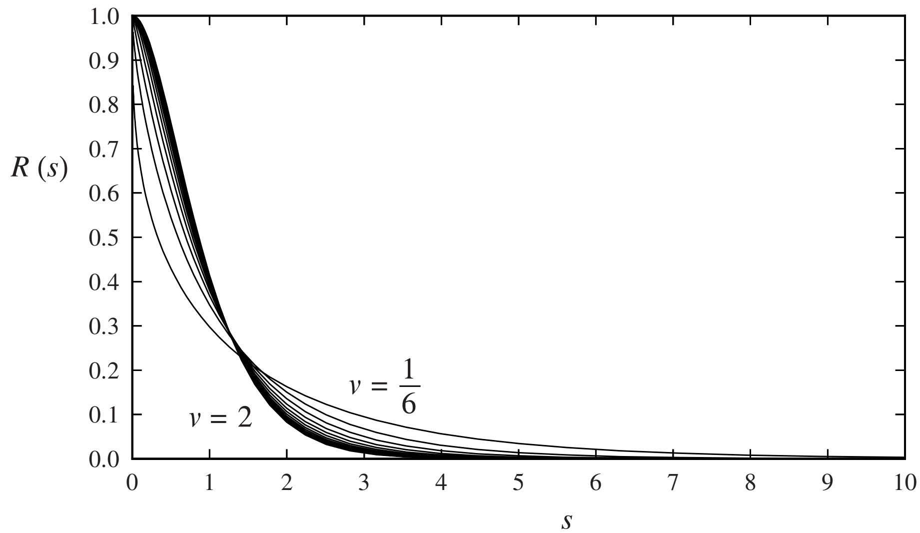


Fig. G.2. Autocorrelation functions $R(s)$, Eq. (G.19), for $\nu = \frac{1}{6}, \frac{1}{3}, \ldots, 1\frac{5}{6}, 2$.

where $K_\nu$ is the modified Bessel function of the second kind. The normalization condition $\langle u^2 \rangle = R(0) = 1$ yields

$$\alpha = \sqrt{\pi}\,\Gamma(\nu + \tfrac{1}{2})/\Gamma(\nu), \tag{G.18}$$

so that Eq. (G.17) can be rewritten

$$R(s) = \frac{2}{\Gamma(\nu)}(\tfrac{1}{2}s\alpha)^\nu K_\nu(s\alpha), \tag{G.19}$$

These autocorrelations are shown in Fig. G.2. (For $\nu = \frac{1}{2}$, the autocorrelation given by Eq. (G.19) is simply $R(s) = \exp(-|s|)$.)

The expression for the autocorrelation is far from revealing. However, the expansion for $K_\nu(s\alpha)$ (for small argument) leads to very informative expressions for the structure function (for small $s$):

$$D(s) = \begin{cases} 2\dfrac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(\tfrac{1}{2}\alpha s)^{2\nu}\ldots, & \text{for } \nu < 1, \\ 2(\nu - 1)(\tfrac{1}{2}\alpha s)^2\ldots, & \text{for } \nu > 1. \end{cases} \tag{G.20}$$

Hence the structure function varies as a power-law with exponent

$$q = \begin{cases} 2\nu, & \text{for } \nu < 1, \\ 2, & \text{for } \nu > 1. \end{cases} \tag{G.21}$$

This behavior is evident in Fig. G.3.

The conclusions to be drawn from the expressions for the power-law exponents $p$ and $q$ (Eqs. (G.16) and (G.21)) are straightforward. The case

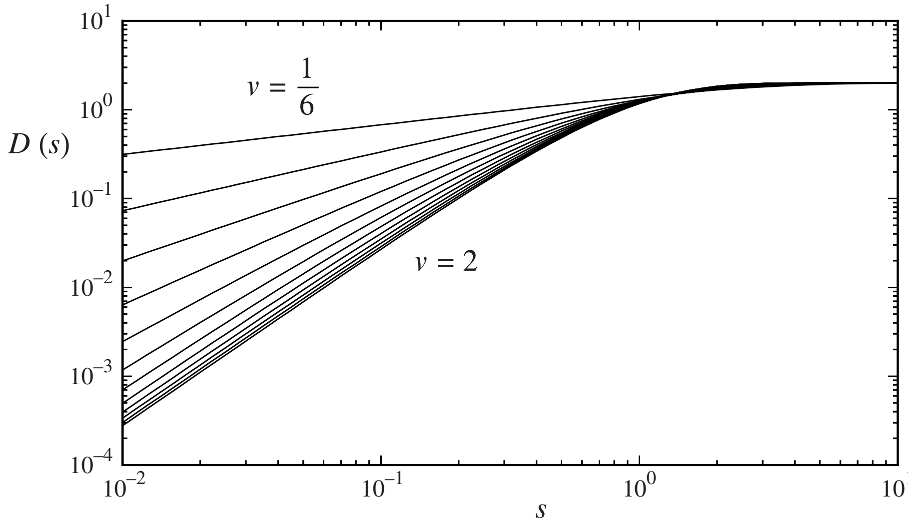


Fig. G.3. Second-order structure functions $D(s)$, Eq. (G.20), for $\nu = \frac{1}{6}, \frac{1}{3}, \ldots, 1\frac{5}{6}, 2$. Observe that, for $\nu > 1$ and small $s$, all the structure functions vary as $s^2$.

Table G.1. *The relationships among the spectral exponent p, the structure-function exponent q, and the differentiability of the underlying process u(t) for the power-law spectrum Eq. (G.14)*

| Parameter in Eq. (G.14) $\nu$ | Spectrum $E(\omega) \sim \omega^{-p}$ $p$ | Structure function $D(s) \sim s^q$ $q$ | Process $u(t)$ $\left\langle \left( \frac{d^n u}{dt^n} \right)^2 \right\rangle < \infty$ $n$ |
|---|---|---|---|
| $\frac{1}{3}$ | $\frac{5}{3}$ | $\frac{2}{3}$ | 0 |
| $\frac{1}{2}$ | 2 | 1 | 0 |
| $> 1$ | $> 3$ | 2 | $\geq 1$ |
| $> 2$ | $> 5$ | 2 | $\geq 2$ |

$\nu > 1$ corresponds to an underlying process that is at least once continuously differentiable, for which $p$ is greater than 3 and $q$ is 2. The case $0 < \nu < 1$ corresponds to a non-differentiable process $u(t)$, and the power laws are connected by

$$p = q + 1. \tag{G.22}$$

Table G.1 displays these results for particular cases.

For $\nu < 1$, for large $\omega$ the power-law spectrum is

$$E(\omega) \approx C_1 \omega^{-(1+q)}, \tag{G.23}$$

while for small $s$ the structure function is

$$D(s) \approx C_2 s^q. \tag{G.24}$$

These relations stem from Eqs. (G.14) and (G.20), from which $C_1$ and $C_2$ (which depend upon $q$) can be deduced. It is a matter of algebra (see Exercise G.1) to show that $C_1$ and $C_2$ are related by

$$\frac{C_1}{C_2} = \frac{1}{\pi}\Gamma(1+q)\sin\left(\frac{\pi q}{2}\right). \tag{G.25}$$

For the particular case $q = \frac{2}{3}$, this ratio is 0.2489; or, to an excellent approximation,

$$(C_1/C_2)_{q=2/3} \approx \tfrac{1}{4}. \tag{G.26}$$

Although we have considered a specific example of a power-law spectrum (i.e., Eq. (G.14)), the conclusions drawn are general. If a spectrum exhibits the power-law behavior $E(\omega) \approx C_1\omega^{-p}$ over a significant range of frequencies, then there is corresponding power-law behavior $D(s) \approx C_2 s^q$ for the structure function with $q = \min(p-1, 2)$. (This assertion can be verified by analysis of Eq. (G.4), see Monin and Yaglom (1975).) For $q < 2$, $C_1$ and $C_2$ are related by Eq. (G.25).

### EXERCISE

G.1 Identify $C_1$ and $C_2$ in Eqs. (G.23) and (G.24). With the use of the following properties of the gamma function:

$$\Gamma(1+v) = v\Gamma(v), \tag{G.27}$$

$$\Gamma(v)\Gamma(1-v) = \pi/\sin(\pi v), \tag{G.28}$$

$$\Gamma(v)\Gamma(v+\tfrac{1}{2}) = (2\pi)^{\frac{1}{2}}2^{(1/2-2v)}\Gamma(2v), \tag{G.29}$$

verify Eq. (G.25).