A proposed modification of the Germano subgrid-scale closure method

D. K. Lilly
University of Oklahoma, CAPS, Energy Center 1110, Norman, Oklahoma 73019

(Received 2 July 1991; accepted 5 November 1991)

The subgrid-scale closure method developed by Germano *et al.* [Phys. Fluids A **3**, 1760 (1991)] is modified by use of a least squares technique to minimize the difference between the closure assumption and the resolved stresses. This modification removes a source of singularity and is believed to improve the method's applicability.

Recently Germano *et al.*[1] subsequently designated G4, developed a new subgrid-scale (SGS) closure that appears to offer marked advantages over the widely used Smagorinsky,[2] hence S, closure. The S closure was first applied extensively to three-dimensional turbulence simulations by Deardorff.[3] Lilly[4] had earlier evaluated the necessary dimensionless coefficient for the S model, based on the assumption that the grid scale lies within an isotropic and homogeneous inertial range of turbulence. As described by Deardorff,[5] Lilly's value of the coefficient worked well when applied to turbulence produced by buoyant instability. For shear-driven turbulence, however, Deardorff and others found it necessary to use a smaller coefficient. These discrepancies have been verified by other investigators, but the reasons remain somewhat obscure.

The G4 closure assumes use of the S formulation, but allows for temporal and spatial variability of the coefficient. It is determined by evaluating the stress–strain relationship at scales of motion a little larger than the grid scale, for which the stresses are explicitly resolved. In the present analysis, we follow G4 and also Moin *et al.*,[6] who extend the G4 analysis to compressible flow and advection of a passive scalar. A potentially important modification is introduced, by which the stress–strain relationship is optimized with a least squares approach. This also removes or reduces a singularity problem in the G4 formulation.

We assume incompressible Boussinesq dynamics. The tensor equation of motion for variables that have been spatially averaged or filtered on the scale of their spatial resolution is given by

$$\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{\pi}}{\partial x_i}=\frac{\partial \tau_{ij}}{\partial x_j}, \tag{1}$$

where the overbar represents the spatial filtering, hence called the grid-scale filter. The variable π is pressure divided by a reference density, while the SGS stress tensor τ_{ij} is defined by

$$\tau_{ij}=(\bar{u}_i\bar{u}_j-\overline{u_iu_j}). \tag{2}$$

Viscous terms have been neglected. Incompressible continuity has been assumed for the grid-filtered variables, so that

$$\frac{\partial \bar{u}_i}{\partial x_i}=0. \tag{3}$$

A second, coarser spatial filter, called the "test" filter, is now applied, and signified by a caret over the overbar. The test-filtered equations of motion are written as

$$\frac{\partial \hat{\bar{u}}_i}{\partial t}+\hat{\bar{u}}_j\frac{\partial \hat{\bar{u}}_i}{\partial x_j}+\frac{\partial \hat{\bar{\pi}}}{\partial x_i}=\frac{\partial}{\partial x_j}(\hat{\bar{u}}_i\hat{\bar{u}}_j-\widehat{\overline{u_iu_j}})=\frac{\partial T_{ij}}{\partial x_j}. \tag{4}$$

The test-filtered continuity equation is similar to (3).

The S closure, applied to the SGS stress defined in (2), is given by

$$\tau_{ij}-\tfrac{1}{3}\delta_{ij}\tau_{kk}=2C\Delta^2|\bar{S}|\bar{S}_{ij}, \tag{5}$$

where $\delta_{ij}=1$ if $i=j$ and zero otherwise, $\bar{S}_{ij}=(\partial\bar{u}_i/\partial x_j+\partial\bar{u}_j/\partial x_i)/2$, and $|\bar{S}|=(2\bar{S}_{kl}\bar{S}_{kl})^{1/2}$. The quantity C is the Smagorinsky coefficient (actually the square of the original quantity), and Δ is the grid filter scale, typically equal to the grid spacing. The second term on the lhs assures that, in the absence of shear, the stress tensor is isotropic, with its trace equal to minus twice the subgrid-scale kinetic energy. The subtest-scale (STS) stress T_{ij} is similarly approximated by

$$T_{ij}-\tfrac{1}{3}\delta_{ij}T_{kk}=2C\hat{\Delta}^2|\hat{\bar{S}}|\hat{\bar{S}}_{ij} \tag{6}$$

with the test-scale shears defined similarly to those for the grid scale. The test filter scale is signified by $\hat{\Delta}$.

The major insight to the SGS problem contributed by G4 is the recognition that consistency between (5) and (6) depends on a proper local choice of C. This is shown by subtraction of the test-scale average of τ_{ij} from T_{ij} to obtain

$$L_{ij}=T_{ij}-\hat{\tau}_{ij}=-\widehat{\bar{u}_i\bar{u}_j}+\hat{\bar{u}}_i\hat{\bar{u}}_j. \tag{7}$$

The elements of L are the resolved components of the stress tensor associated with scales of motion between the test scale and the grid scale. We will call these scales the "test window." The test-window stresses, the rhs of (7), can be explicitly evaluated and compared locally with the S closure approximation by subtracting the test-scale average of (5) from (6), i.e.,

$$L_{ij}-\tfrac{1}{3}\delta_{ij}L_{kk}=2CM_{ij}, \tag{8}$$

where

$$M_{ij}=\hat{\Delta}^2|\hat{\bar{S}}|\hat{\bar{S}}_{ij}-\Delta^2\widehat{|\bar{S}|\bar{S}_{ij}}. \tag{9}$$

One now seeks the value of C that solves (8) and then applies that value to (5). Since (8) represents five independent equations in one unknown, no value of C can be chosen to make it correct. Its error can be minimized by

 0899-8213/92/030633-03$04.00

applying a least squares approach. Define Q to be the square of the error in (8), i.e.,

$$Q=(L_{ij}-\tfrac{1}{3}\delta_{ij}L_{kk}-2CM_{ij})^2. \tag{10}$$

Upon setting $\partial Q/\partial C=0$, C is evaluated as

$$C=\tfrac{1}{2}(L_{ij}M_{ij}/M_{ij}^2). \tag{11}$$

This represents the minimum of Q, since it is easily shown that $\partial^2 Q/\partial C^2>0$. Note that the isotropizing term in (8) and (10) does not appear in the numerator of (11) because $\bar{S}_{ii}=0$ in an incompressible flow.

The above evaluation of C differs from that of G4, who contract (8) by multiplying both sides by $\bar{S}_{ij}$ to obtain

$$C=\tfrac{1}{2}(L_{ij}\bar{S}_{ij}/M_{ij}\bar{S}_{ij}). \tag{12}$$

By this process, they equate one of the many possible projections of (8). The physical meaning of (12) is not obvious, although it is dimensionally similar to equating the rates of energy dissipation from the grid and test scales. In tests using data from direct simulations, G4 found that the denominator of (12) could vanish or become so small as to lead to a computationally unstable value of C. To avoid this problem they averaged the numerator and denominator over planes parallel to the lower boundary, thereby perhaps losing some of the conceptual advantages of their formulation. By contrast, the denominator of (11) can vanish only if each of the five independent components of M_{ij} vanish separately, that is if the test scale strain vanishes completely. In that case, the numerator vanishes also.

Moin[7] reports that (11) has now been tested in simulations with apparently favorable results. He states, however, that, if the method is applied to individual grid points, the value of C still becomes large enough occasionally to lead to computational instability. We have carried out (but do not show here) a brief statistical analysis of (11), assuming that L_{ij} and M_{ij} are Gaussian variables. It is found that the variance of C is proportional to $(\sigma_L/\sigma_M)^2$, where σ_L and σ_M are the standard deviations of the L's and M's. The proportionality constant varies inversely with the number of degrees of freedom, suggesting that some averaging may be necessary to avoid excessively large values. Alternatively, isolated large values of C may simply be truncated.

The G4 method, with the above modification, appears to be a plausible solution to an old but previously unrealized goal of large eddy simulation, which is to improve the SGS closure by "tuning" it to match the statistical structure of resolved turbulent eddies. The numerator of (11), and therefore the sign of C, can locally become negative, leading to "backscatter," i.e., transfer of energy upscale. This is regarded by G4 as a favorable aspect of their closure method, since it recognizes the ability of the subgrid scales to add randomness to the explicit scales. By comparing large eddy simulations with direct simulations of higher resolution, G4 found that their stresses are much better correlated with the direct-simulation stresses than are those of the S closure. G4 found the optimum ratio of the test and grid scales to be $\hat{\Delta}/\Delta=2$. Thus the components of L and M occupy the highest resolvable octave in wave space.

Moin *et al.*[6] show an extension of the G4 analysis to a compressible gas and to SGS flux of temperature. They also show a method for determining an optimal local value of the turbulent Prandtl number, and by extension the viscosity–diffusivity ratio for any conserved scalar. We outline their scalar transport analysis here in simplified Boussinesq form, and again solve with a least squares technique. Temperature is assumed to be a conserved variable. The evolution of the grid-filtered temperature is therefore given by

$$\frac{\partial \bar{T}}{\partial t}+\bar{u}_j\frac{\partial \bar{T}}{\partial x_j}=\frac{\partial}{\partial x_j}(\bar{u}_j\bar{T}-\overline{u_jT})=\frac{\partial h_j}{\partial x_j}. \tag{13}$$

Similarly the test-filtered temperature evolves according to

$$\frac{\partial \hat{\bar{T}}}{\partial t}+\bar{u}_j\frac{\partial \hat{\bar{T}}}{\partial x_j}=\frac{\partial}{\partial x_j}(\hat{\bar{u}}_j\hat{\bar{T}}-\widehat{\overline{u_jT}})=\frac{\partial H_j}{\partial x_j}, \tag{14}$$

where h_j and H_j are SGS and STS temperature fluxes, respectively. The S closure is now applied to h_j and H_j, i.e.,

$$h_j=\frac{2C\Delta^2}{\mathrm{Pr}}|\bar{S}|\frac{\partial \bar{T}}{\partial x_j} \tag{15}$$

and

$$H_j=\frac{2C\hat{\Delta}^2}{\mathrm{Pr}}|\hat{\bar{S}}|\frac{\partial \hat{\bar{T}}}{\partial x_j}, \tag{16}$$

where the eddy Prandtl number Pr remains to be determined. Upon defining the test-window temperature fluxes, $P_j=H_j-h_j=\widehat{\bar{T}\,\bar{u}_j}-\hat{\bar{T}}\hat{\bar{u}}_j$, a least squares procedure is again applied, leading to the Prandtl number prediction,

$$\frac{1}{\mathrm{Pr}}=\frac{1}{2C}\frac{P_jR_j}{R_j^2}=\frac{M_{ik}^2}{L_{ik}M_{ik}}\frac{P_jR_j}{R_j^2}, \tag{17}$$

where (11) has been used to obtain the second equality, and

$$R_j=\hat{\Delta}^2|\hat{\bar{S}}|\widehat{\frac{\partial \bar{T}}{\partial x_j}}-\Delta^2\widehat{|\bar{S}|\frac{\partial \bar{T}}{\partial x_j}}. \tag{18}$$

Again (17) involves only explicit variables and is likely to be well behaved.

ACKNOWLEDGMENTS

I am indebted to P. Moin, M. Germano, and especially Z. Kogan for helpful comments.

This work was supported by the National Science Foundation through Grants No. ATM-8914434 and No. 8809862, the latter to the Center for the Analysis and Prediction of Storms.

[1] M. Germano, U. Piomelli, P. Moin, and W. H. Cabot, "A dynamic subgrid-scale eddy viscosity model," Phys. Fluids A 3, 1760 (1991).

[2] J. Smagorinsky, "General circulation experiments with the primitive equations, I. The basic experiment," Mon. Weather Rev. **91**, 99 (1963).

[3] J. W. Deardorff, "A three-dimensional numerical study of turbulent channel flow at large Reynolds numbers," J. Fluid Mech. **41**, 453 (1970).

[4]D. K. Lilly, "On the application of the eddy viscosity concept in the inertial sub-range of turbulence," NCAR Manuscript No. 123, National Center for Atmospheric Research, Boulder, CO, 1966.
[5]J. W. Deardorff, "On the magnitude of the subgrid scale eddy coefficient," J. Comput. Phys. 7, 120 (1971).
[6]P. Moin, K. Squires, W. Cabot, and S. Lee, "A dynamic subgrid-scale model for compressible turbulence and scalar transport," Phys. Fluids A 3, 2746 (1991).
[7]P. Moin (private communication).