# Grid-point requirements for large eddy simulation: Chapman's estimates revisited

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## 1. Motivation and objectives

In his 1979 landmark paper, Chapman (1979) estimated the required numbers of grid points for large eddy simulation (LES) of turbulent boundary layers with and without wall modeling, and emphasized the importance of wall modeling for LES of practical flows of aeronautical interest. According to his calculations, the number of grid points ($N$) required for wall-modeled LES is proportional to $Re_{L_x}^{2/5}$, and a wall-resolving LES requires $N \sim Re_{L_x}^{9/5}$, where $Re_{L_x} = UL_x/\nu$, $U$ is the free-stream velocity, $L_x$ is the flat-plate length in the streamwise direction, and $\nu$ is the kinematic viscosity. In arriving at these Reynolds number dependencies Chapman used the following formulae:

$$c_f = 0.045 Re_\delta^{-1/4}, \tag{1.1}$$

$$\theta = \frac{7}{72}\delta, \tag{1.2}$$

where $c_f$ is the wall skin friction coefficient, $Re_\delta = U\delta/\nu$ is the Reynolds number based on the boundary layer thickness ($\delta$) and the free-stream velocity, and $\theta$ is the momentum thickness. Eq. (1.1) is the correlation of the skin friction coefficient with the Reynolds number for circular pipe flow, and Eq. (1.2) is from the seventh-power velocity distribution law (Schlichting 1955). From Eqs. (1.1) and (1.2) and $c_f = 2d\theta/dx$, one can easily show that for spatially evolving turbulent boundary layers

$$\frac{\delta}{x} = 0.37 Re_x^{-1/5}, \tag{1.3}$$

$$c_f = 0.0577 Re_x^{-1/5}, \tag{1.4}$$

where $Re_x = Ux/\nu$, and $x$ is the streamwise distance. Using Eqs. (1.3) and (1.4), Chapman (1979) showed that the required numbers of grid points for LES with and without wall modeling are $N_{wm} \sim Re_{L_x}^{2/5}$ and $N_{wr} \sim Re_{L_x}^{9/5}$, respectively. The weak point in this derivation is that Eq. (1.4) is valid for low to intermediate Reynolds number range (i.e., $Re_x \leq 10^6$) and shows a significant deviation from experimental measurements at higher Reynolds numbers (White 2005; Nagib *et al.* 2007). Also, the momentum thickness obtained from Eq. (1.3), $\theta/x = 0.036 Re_x^{-1/5}$, does not fit the experimental data in the literature (Monkewitz *et al.* 2007).

In the present study, we use a more accurate formula for high Reynolds number boundary layer flow to suggest new grid-point requirements for wall-modeled and wall-resolving LES. A power-law curve-fit approximation of the skin friction coefficient for high Reynolds number range ($10^6 \leq Re_x \leq 10^9$) is (White 2005)

$$c_f = 0.020 Re_\delta^{-1/6}. \tag{1.5}$$

Again with the seventh-power velocity distribution, we obtain

$$\frac{\delta}{x} = 0.16 Re_x^{-1/7}, \tag{1.6}$$

$$c_f = 0.027 Re_x^{-1/7}. \tag{1.7}$$

A recent study by Nagib *et al.* (2007) for high Reynolds number boundary layer flow supports this Reynolds number dependence of $c_f$ with a correction of the coefficient from 0.027 to 0.02358 for a better fit to the experimental data. Also, Monkewitz *et al.* (2007) showed that a power-law curve-fit of the momentum thickness, $\theta/x = 0.016 Re_x^{-0.15}$, agrees very well with the experimental data in $10^6 \leq Re_x \leq 10^9$, which is not very different from the present one ($\theta/x = 0.0156 Re_x^{-1/7}$ from Eqs. (1.2) and (1.6)). Accordingly, Eqs. (1.6) and (1.7) are used to obtain new grid-point requirements for wall-modeled and wall-resolving LES.

## 2. Grid point requirements for LES and DNS

In this section, we present the numbers of grid points required for wall-modeled LES, wall-resolved LES, and direct numerical simulation (DNS).

### 2.1. *Wall-modeled LES*

We determine the required number of grid points for wall-modeled LES. Chapman (1979) considered a cubic computational box with side length equal to the average boundary layer thickness, $\bar{\delta}$, as defined by $\bar{\delta} = \frac{1}{L_x}\int_0^{L_x} \delta(x)dx$, and distributed grid points uniformly in the streamwise, wall-normal ($y$) and spanwise ($z$) directions. Then, he showed that the total number of grid points required for wall-modeled LES is $N_{wm} \sim Re_{L_x}^{2/5}$. When we follow the same procedure as in Chapman (1979) but use Eq. (1.6) for high Reynolds number boundary layer flow, the total number of grid points for the entire computational domain of dimensions $L_x \times \delta(L_x) \times L_z$ becomes $N_{wm} \sim Re_{L_x}^{2/7}$.

A more accurate estimation of $N_{wm}$ requires an integration of the boundary layer thickness in the streamwise direction (rather than taking the average boundary layer thickness) (Spalart *et al.* 1997). At a given streamwise location $x$, we consider the grid spacings of $\Delta x = \delta/n_x, \Delta y = \delta/n_y$, and $\Delta z = \delta/n_z$ in the streamwise, wall-normal ($y$) and spanwise ($z$) directions, respectively. Then, for the entire computational domain of dimensions $L_x \times \delta \times L_z$, the total number of grid points for wall-modeled LES is

$$N_{total} = \int_0^{L_x}\int_0^{L_z} \frac{n_x n_y n_z}{\delta^2}\, dxdz. \tag{2.1}$$

Because Eqs. (1.6) and (1.7) are valid in the high Reynolds number range, Eq. (2.1) can be rewritten as

$$N_{total} = N(x < x_0) \; + \; \int_{x_0}^{L_x}\int_0^{L_z} \frac{n_x n_y n_z}{\delta^2}\, dxdz, \tag{2.2}$$

where $x_0$ is the streamwise location beyond which Eqs. (1.6) and (1.7) are valid. With Eq. (1.6), the second term in Eq. (2.2) becomes

$$N_{wm} = 54.7\, \frac{L_z}{L_x}\; n_x n_y n_z \; Re_{L_x}^{2/7}\left[\left(\frac{Re_{L_x}}{Re_{x_0}}\right)^{5/7} - 1\right], \tag{2.3}$$

indicating that $N_{wm} \sim Re_{L_x}$. Eq. (2.3) can be represented in terms of different Reynolds number definitions: $N_{wm} = 101\ (L_z/L_x)\ n_x n_y n_z\ Re_{\delta_{L_x}}^{1/3}\ \left[\left(Re_{\delta_{L_x}}/Re_{\delta_{x_0}}\right)^{5/6} - 1\right] = 233\ (L_z/L_x)\ n_x n_y n_z\ Re_{\tau_{L_x}}^{4/11}\ \left[\left(Re_{\tau_{L_x}}/Re_{\tau_{x_0}}\right)^{10/11} - 1\right]$, where $Re_{\delta_x} = U\delta(x)/\nu$, $Re_{\tau_x} = u_\tau(x)\ \delta(x)/\nu$, $u_\tau(= \sqrt{\tau_w/\rho})$ is the wall shear velocity, $\tau_w$ is the wall shear stress and $\rho$ is the density. Here, $n_x n_y n_z$ is the number of grid points to resolve the cubic computational volume of $\delta^3$ exterior to the viscous wall region. Chapman (1979) suggested $n_x n_y n_z = 2500$, corresponding to $n_x = 10, n_y = 25$ and $n_z = 10$. In most wall-modeled LES to date the number of grid points $n_x n_y n_z$ used to resolve the cubic computational volume of $\delta^3$ has been in the range $1200 \sim 33000$ ($n_x = 5 \sim 32, n_y = 16 \sim 32$ and $n_z = 15 \sim 32$) (Spalart *et al.* 1997; Cabot & Moin 1999; Nicoud *et al.* 2001; Piomelli & Balaras 2002; Keating & Piomelli 2006; Pantano *et al.* 2008; Davidson 2009; Kawai & Larsson 2010).

### 2.2. *Wall-resolved LES*

In this subsection, we estimate the required number of grid points for wall-resolving LES, following the procedure used by Chapman (1979). Let us consider a small computational box of dimensions $dx \times l_y \times dz$ within the viscous wall region, where $l_y^+ = l_y u_\tau/\nu \approx 100$. Uniform grid spacings in the streamwise ($dx$) and spanwise ($dz$) directions to resolve the wall region are denoted as $\Delta x_w$ and $\Delta z_w$, respectively. Then, the number of grid points in the computational box is

$$\Delta N = \frac{dx}{\Delta x_w}\ n_y\ \frac{dz}{\Delta z_w} = \frac{n_y}{\Delta x_w^+ \Delta z_w^+}\ \frac{\tau_w\ dxdz}{\rho\nu^2}, \tag{2.4}$$

where $n_y$ is the number of grid points stretched in the wall-normal direction, $0 \le y^+ \le l_y^+$. The total number of grid points for the entire near-wall computational domain is obtained by integrating Eq. (2.4) over the domain:

$$N_{wr} = \frac{4}{3}\frac{n_y}{\Delta x_w^+ \Delta z_w^+}\ \frac{\int_{x_0}^{L_x}\int_0^{L_z} \tau_w dz dx}{\rho\nu^2}. \tag{2.5}$$

Here, as suggested by Chapman (1979), multiple blocks of nested grids with $\Delta x$ and $\Delta z$ doubling outward from block to block are used to relieve the excessive grid point requirement; without using nested grids, the required number of grid points would increase by about a factor of 10. With Eq. (1.7), the required number of grid points for wall-resolving LES is

$$N_{wr} = 0.021\frac{n_y}{\Delta x_w^+ \Delta z_w^+}\ \frac{L_z}{L_x}\ Re_{L_x}^{13/7}\left[1 - \left(\frac{Re_{x_0}}{Re_{L_x}}\right)^{6/7}\right], \tag{2.6}$$

showing $N_{wr} \sim Re_{L_x}^{13/7}$ rather than $Re_{L_x}^{9/5}$ estimated by Chapman (1979). In terms of different Reynolds number definitions, $N_{wr} = 1.11\ n_y/(\Delta x_w^+ \Delta z_w^+)\ \ (L_z/L_x)\ Re_{\delta_{L_x}}^{13/6}\ \left(1 - Re_{\delta_{x_0}}/Re_{\delta_{L_x}}\right) = 259\ n_y/(\Delta x_w^+ \Delta z_w^+)\ (L_z/L_x)\ Re_{\tau_{L_x}}^{26/11}\ \left[1 - \left(Re_{\tau_{x_0}}/Re_{\tau_{L_x}}\right)^{12/11}\right]$. Chapman suggested $\Delta x_w^+ \approx 100$, $\Delta z_w^+ \approx 20$ and $n_y \approx 10$. Typically, $\Delta x_w^+ \approx 50 \sim 130$, $\Delta z_w^+ \approx 15 \sim 30$ and $n_y \approx 10 \sim 30$ have been used in wall-resolving LES (see, for example, Kravchenko *et al.* 1996).

### 2.3. *DNS*

The required number of grid points for DNS is proportional to $Re^{9/4}$ (Rogallo & Moin 1984), where the grid spacings should be sufficiently fine to resolve the dissipation length scale. However, in this estimate, the Reynolds number is defined in terms of large eddy characteristic velocity and length scales. We derive the required number of grid points for DNS based on the streamwise length $L_x$, as was done for wall-modeled and wall-resolving LES.

Let us consider a small computational box of dimensions $dx \times dy \times dz$ inside the boundary layer. The Kolmogorov length scale $\eta$ $(= (\nu^3/\epsilon)^{1/4})$ is estimated from the dissipation rate in a turbulent boundary layer, $\epsilon = \nu \overline{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}}$, where the overline denotes time averaging. The dissipation rate is largest at the wall, i.e.,

$$\epsilon_{max} = \nu \left( \frac{\partial \bar{u}}{\partial y}\frac{\partial \bar{u}}{\partial y} + \overline{\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}} + \overline{\frac{\partial w'}{\partial y}\frac{\partial w'}{\partial y}} \right) \text{ at the wall,} \tag{2.7}$$

and decays rapidly away from the wall. Here, $u$ and $w$ are the streamwise and spanwise velocities, respectively, and the prime denotes the velocity fluctuations. Note that we include the mean velocity gradient term in Eq. (2.7) because the main concern of the present analysis is the grid resolution near the wall. Among the three terms on the right hand side of Eq. (2.7), the first term is largest: the ratios of second and third terms to the first one are much less than 1 (e.g., the ratio of second to first terms is about 0.16), but they increase weakly with the Reynolds number (Hu *et al.* 2006; Örlü & Schlatter 2011). Therefore, we take the first term in Eq. (2.7) and use Eq. (1.7) to estimate the Kolmogorov length scale in turbulent boundary layer. When uniformly spaced grids are used along $dx$, the number of grid points to resolve the Kolmogorov length scale is

$$\Delta N_x = \frac{dx}{\eta} = \frac{dx}{x}\frac{x}{\eta} = 0.116\ \frac{dx}{x} Re_x^{13/14}. \tag{2.8}$$

Then, the total number of grid points required for DNS in this small computational box is

$$\Delta N = \frac{dx}{\eta}\ \frac{dy}{\eta}\ \frac{dz}{\eta} = 0.00157\ \frac{dx}{x}\frac{dy}{x}\frac{dz}{x} Re_x^{39/14}. \tag{2.9}$$

The total number of grid points for the entire computational domain is obtained by integrating Eq. (2.9) over the domain $(L_x \times \delta \times L_z)$: i.e.,

$$N_{DNS} = 0.00157 \int_{x_0}^{L_x} \int_0^{\delta(x)} \int_0^{L_z} Re_x^{39/14} \frac{1}{x^3}\ dz dy dx = 0.00157 L_z \int_{x_0}^{L_x} Re_x^{39/14} \frac{\delta}{x^3}\ dx. \tag{2.10}$$

Using Eq. (1.6), we finally obtain

$$N_{DNS} = 0.000153\ \frac{L_z}{L_x}\ Re_{L_x}^{37/14} \left[ 1 - \left( \frac{Re_{x_0}}{Re_{L_x}} \right)^{23/14} \right], \tag{2.11}$$

showing that $N_{DNS} \sim Re_{L_x}^{37/14}$ rather than $Re_{L_x}^{9/4}$. Eq. (2.11) can be reformulated in terms of different Reynolds number definitions: $N_{DNS} = 0.0434\ (L_z/L_x)\ Re_{\delta_{L_x}}^{37/12} \left[1 - \left(Re_{\delta_{x_0}}/Re_{\delta_{L_x}}\right)^{23/12}\right] = 101\ (L_z/L_x)\ Re_{\tau_{L_x}}^{37/11} \left[1 - \left(Re_{\tau_{x_0}}/Re_{\tau_{L_x}}\right)^{23/11}\right]$. Note that, in our derivation of Eq. (2.11), we have not considered using nested and/or stretched grids. To use these grids, more information on the spatial development of the turbulent dissipation rate is required.

TABLE 1. Number of grid points required for numerical simulation of flow over a flat-plate airfoil (with an aspect ratio of 4) using the wall-modeled and wall-resolving LES. $Re_{x_0} = 5 \times 10^5$, $n_x n_y n_z = 2500$ and $n_y / \left(\Delta x_w^+ \Delta z_w^+\right) = 1/200$ in Eqs. (2.3) and (2.6) are used. Here, $Re_c = Uc/\nu$ and $c$ is the chord length. Note that in this estimate we do not include the number of grid points required for $x < x_0$.

| $Re_c$ | $N_{wm}$ (wall-modeled LES) | $N_{wr}$ (wall-resolved LES) |
|---|---|---|
| $10^6$ | $3.63 \times 10^7$ | $5.23 \times 10^7$ |
| $10^7$ | $8.20 \times 10^8$ | $7.76 \times 10^9$ |
| $10^8$ | $9.09 \times 10^9$ | $5.98 \times 10^{11}$ |
| $10^9$ | $9.26 \times 10^{10}$ | $4.34 \times 10^{13}$ |

## 3. Conclusions

In this paper, we revisited Chapman's estimate of the numbers of grid points required to simulate turbulent flow above a flat plate of streamwise length $L_x$ using wall-modeled and wall-resolving LES. Using accurate correlations of the skin friction coefficient and boundary layer thickness for high Reynolds number boundary layer flow, we showed that the wall-modeled LES requires $N_{wm} \sim Re_{L_x}$, but the wall-resolving LES requires $N_{wr} \sim Re_{L_x}^{13/7}$. On the other hand, DNS requires $N_{DNS} \sim Re_{L_x}^{37/14}$. The approximate numbers of grid points required for the flow over a flat-plate airfoil using LES with and without modeling the viscous wall region are estimated in Table 1. As shown, the number of grid points for the wall-modeled LES is one to three orders of magnitude smaller than that for the wall-resolving LES, indicating the practical importance of wall modeling in LES for high Reynolds number flows.

Lastly, we present the numbers of grid points required to simulate turbulent boundary layer flow at a local Reynolds number $Re_\delta$ using LES and DNS. For this purpose, we consider a computational domain of dimensions $\alpha\delta \times \delta \times \beta\delta$ in $x \times y \times z$ directions, where $\alpha$ and $\beta$ are determined to include large-scale structures inside the boundary layer. Then, the number of grid points required for the wall-modeled LES is $N_{wm} = \alpha\beta\ n_x n_y n_z$, indicating that $N_{wm}$ is independent of the Reynolds number. In the case of wall-resolved LES, the number of grid points is obtained similarly as in Eq. (2.4): $N_{wr} = (4/3)\ n_y/(\Delta x_w^+ \Delta z_w^+)\ \tau_w/(\rho\nu^2)\ (\alpha\delta)(\beta\delta) = 0.0133\,\alpha\beta\ n_y/(\Delta x_w^+ \Delta z_w^+)\ Re_\delta^{11/6}$. The number of grid points for DNS is also obtained as in Eq. (2.9): $N_{DNS} = (\alpha\delta/\eta)(\delta/\eta)(\beta\delta/\eta) = 0.00099\ \alpha\beta\ Re_\delta^{33/12}$.

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REFERENCES

CABOT, W. & MOIN, P. 1999 Approximate wall boundary conditions in the large-eddy simulation of high Reynolds number flow. *Flow, Turb. & Combust.* **63**, 269–291.

Chapman, D. R. 1979 Computational aerodynamics development and outlook. *AIAA J.* **17** (12), 1293–1313.

Davidson, L. 2009 Large eddy simulations: how to evaluate resolution. *Int. J. Heat Fluid Flow* **30** (5), 1016–1025.

Hu, Z. W., Morfey, C. L. & Sandham, N. D. 2006 Wall pressure and shear stress spectra from direct simulations of channel flow. *AIAA J.* **44** (7), 1541–1549.

Kawai, S. & Larsson, J. 2010 A dynamic wall model for large-eddy simulation of high Reynolds number compressible flows. In *Annu. Res. Briefs - 2010*, pp. 25–37. Center for Turbulence Research, Stanford University, CA.

Keating, A. & Piomelli, U. 2006 A dynamic stochastic forcing method as a wall-layer model for large-eddy simulation. *J. Turb.* **7** (12), 1–24.

Kravchenko, A. G., Moin, P. & Moser, R. 1996 Zonal embedded grids for numerical simulations of wall-bounded turbulent flows. *J. Comput. Phys.* **127** (2), 412–423.

Monkewitz, P. A., Chauhan, K. A. & Nagib, H. M. 2007 Self-consistent high-Reynolds-number asymptotics for zero-pressure-gradient turbulent boundary layers. *Phys. Fluids* **19**, 115101.

Nagib, H. M., Chauhan, K. A. & Monkewitz, P. A. 2007 Approach to an asymptotic state for zero pressure gradient turbulent boundary layers. *Phil. Trans. R. Soc. A* **365** (1852), 755–770.

Nicoud, F., Baggett, J. S., Moin, P. & Cabot, W. 2001 Large eddy simulation wall-modeling based on suboptimal control theory and linear stochastic estimation. *Phys. Fluids* **13** (10), 2968–2984.

Örlü, R. & Schlatter, P. 2011 On the fluctuating wall-shear stress in zero pressure-gradient turbulent boundary layer flows. *Phys. Fluids* **23** (2), 021704.

Pantano, C., Pullin, D. I., Dimotakis, P. E. & Matheou, G. 2008 LES approach for high Reynolds number wall-bounded flows with application to turbulent channel flow. *J. Comput. Phys.* **227** (21), 9271–9291.

Piomelli, U. & Balaras, E. 2002 Wall-layer models for large-eddy simulations. *Annu. Rev. Fluid Mech.* **34**, 349–374.

Rogallo, R. S. & Moin, P. 1984 Numerical simulation of turbulent flows. *Annu. Rev. Fluid Mech.* **16**, 99–137.

Schlichting, H. 1955 *Boundary-layer Theory*. McGraw-Hill.

Spalart, P. R., Jou, W.-H., Strelets, M. & Allmaras, S. R. 1997 Comments on the feasibility of LES for wings, and on a hybrid RANS/KES approach. In *Advances in DNS/LES*. Greyden Press.

White, F. M. 2005 *Viscous Fluid Flow*, 3rd edn. McGraw-Hill.