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IMPROVED SUBGRID-SCALE MODELS FOR LARGE-EDDY SIMULATION

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Abstract

Models for subgrid-scale turbulence are analyzed. The analysis indicates that there is enough information in the resolved scales to allow some of the characteristics of the complete flow field to be determined. The kinetic energy of the small-scale motions can be decomposed into two parts. One is due to energy transfer from the large scales and is correlated with them; the other is uncorrelated. This leads to a two-component *eddy-viscosity model*. The "*production equals dissipation*" argument does not hold for the small scales in the decay of turbulence because it does not account for the uncorrelated component. The two-component model can be reduced to the single-component models that have been used previously, but it shows some of the flaws in arguments made earlier and explains some of the discrepancies that have been observed.

The exchange between the large and small scales takes place mainly between the smallest scales of the former and the largest scales of the latter. This argument is the basis of a new model *which preliminary tests show to be superior to the* Smagorinsky model that has been used heretofore. Finally, a new length scale for use with anisotropic filters is proposed.

I. Introduction

Turbulent flows contain structures of various length scales. The large-scale motions contain most of the energy, are anisotropic, and do most of the transporting, while the small-scale motions are mainly dissipative. Present computer capabilities do not allow computation of all scales of motion, except for very low Reynolds numbers. Large-eddy *simulation (LES) attempts to compute the large* scales and model the small ones (the so-called subgrid scale or SGS motions) at higher Reynolds numbers.

The simplest SGS models assume that the SGS Reynolds stress tensor is proportional to the stress tensor, S_{ij}, of the large-scale field. The proportionality factor is the SGS eddy viscosity ν_T. In particular, the Smagorinsky model (Smagorinsky, 1963) assumes the eddy viscosity is proportional to $\Delta^2|S|$ while the vorticity model assumes it proportional to $\Delta^2|\omega|$. These models are currently used by the Stanford and Queen Mary College groups, as stated by Ferziger and Leslie (1979). Some unresolved issues on SGS modeling have been presented by Herring (1977) and/or Ferziger and Leslie (1979). The chief of these follow.

"Production Equals Dissipation"

This argument assumes that the rate of transfer of energy from the resolved motions to the small scales is equal to the rate of dissipation in the small scales and can be used to derive the above models.

This argument is not valid in the decay of turbulence, because it neglects the decay of the energy initially in the small scales. However, the Smagorinsky and vorticity models have been successfully used by Kwak et al. (1975) and Shaanan et al. (1975), among others, to simulate the (filtered) decay of homogeneous isotropic turbulence experiments of Comte-Bellot and Corrsin (1971).

Velocity Scale

Most eddy-viscosity models assume $\nu_T \sim q\Delta$ and thus require a velocity scale q of the small-scale motions. The Smagorinsky model further assumes $q \sim (\Delta \bar{S})$. McMillan and Ferziger (1979) found that this relation is not very accurate. On the other hand, the model has been used successfully.

Smagorinsky Constant

The constant in Smagorinsky's eddy viscosity model has been determined by Lilly's (1967) theoretical argument as ~ 0.2. Similar values were found by Clark et al. (1977) through a "complete" simulation of a low Reynolds number flow, and by Kwak et al. (1977) and Shaanan et al. (1977) by fitting the (filtered) experimental decay of turbulence. On the other hand, Deardorff (1970) and Schumann (1975) found that this value of the parameter damped too much energy in the simulation of a channel flow. Their empirical results led to a constant of 0.1 for the Smagorinsky model.

Defiltering

An approach to LES which defines the large *scales by spatial filtering is presented in* Appendix A. This approach is currently used by the Stanford, NASA-Ames, and Queen Mary College groups. A "defiltering" process which could produce the characteristics of the flow from LES has not yet been presented. Such a process would improve our understanding of SGS models and would also allow comparison with actual experimental results.

The analysis presented below shed some light on these issues. In particular, it does not require the "production equals dissipation" argument; it allows estimation of some of the characteristic scales of the flow field from resolved variables and leads to a "two-component" eddy viscosity model which simplifies to the eddy-viscosity models when $\Delta \ll L$; it explains the poor correlation between q and $|\Delta \bar{S}|$ and explains some of the difference between the values of the Smagorinsky constant *found by various authors. We shall also discuss* length scales for use with anisotropic grids and suggest new subgrid-scale models. The main objective of this work is to reach a better understanding of the simplest SGS models in order to derive better models and thereby to improve the simulation of different turbulent flows.

The equations describing the complete flow field (u) and resolvable or filtered flow field

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$(\bar{u})$ are presented in Appendix B of this work. A simple way to close the filtered equations is through use of an eddy viscosity model. A successful eddy-viscosity model should accurately represent the effects of the small scales in terms of the resolved variables. We shall therefore try to defilter the resolved field first and to model the eddy viscosity afterwards.

The analysis is compared with the turbulence decay experiment of Comte-Bellot and Corrsin (1971); in particular, we use the experimental case with $U_\infty = 10$ m/s, $M = 5.08$ cm, and $R_\lambda \approx 70$. The constants are suitable for this flow but apply to their other cases with only slightly less accuracy.

II. Defiltering

The principal quantities in the decay of turbulence are the turbulence kinetic energy, the dissipation rate, and a suitable length scale; this is the case for both the full and filtered fields. Dimensional analysis for the full and filtered fields at high Reynolds number leads to the relationships presented in Table 1 below. Recall that the "dissipation" in the filtered field is actually the energy transferred to the small-scale field.

As Lilly (1967) suggested, L is the average length scale of the energy-containing eddies. This length scale makes the normalized large-scale spectrum independent of Reynolds number (see Tennekes and Lumley, 1978, p. 267).

The filtered field contains the same large eddies as the full field, so we expect:

$$L_f \sim L \tag{1}$$

Therefore, from the first line of Table 1, we also expect

$$\varepsilon^{2/3} \sim \varepsilon_f^{2/3}\, Q^2/Q_f^2 \tag{2}$$

This last equation tells us that, since filtering reduces the kinetic energy of the flow (i.e., $Q_f^2 < Q^2$), the rate of energy transfer from the resolved scales to the small scales, ε_f, is smaller than the total rate of dissipation, ε (i.e., $\varepsilon_f < \varepsilon$). The difference is due to the decay of the energy which was initially in the small scales and is frequently quite large. One cannot apply the "production equals dissipation" argument to the small scales if this is the case.

The small-scale field is defined by LES as the difference between the full field and the filtered field. The principal quantities for the small scales are its kinetic energy q^2, dissipation rate ε, and the filter width Δ (their natural length scale). Dimensional analysis suggests that, if the Reynolds number is high enough that there is no significant viscous dissipation of eddies of size Δ, then these scales are related as shown in Table 2 below.

Table 2

Properties of the Small-Scale Field

Dissipation	$\varepsilon \sim q^3/\Delta$
Energy Balance	$q^2_{,t} = 2(\varepsilon_f - \varepsilon)$
Kinetic Energy	$q^2 \equiv Q^2 - Q_f^2 = \langle u_i u_i - \bar{u}_i \bar{u}_i \rangle$

The principal objective of this section is to determine the characteristics of the full field from the filtered variables. Writing the first line of Table 2 as the equality

$$q^2 = c_1 (2\Delta\varepsilon)^{2/3} \tag{3}$$

and using the last line of Table 2 and Eqn. (2), we find that the total kinetic energy can be estimated from:

$$Q^2 = \frac{Q_f^4}{Q_f^2 - c_1(2\Delta\varepsilon_f)^{2/3}} \tag{4}$$

where c is a constant of order unity to be determined.

The scaling and defiltering methods proposed above can be tested for the experiment on the decay of homogeneous isotropic turbulence of Comte-Bellot and Corrsin (1971). An exact or direct simulation of this flow at low turbulence Reynolds numbers might provide an even better test of the defiltering process.

Figure 1 presents a test of the dissipation scaling of Eq. (2). Fig. 2 tests the scaling for the subgrid-scale energy presented in Eq. (3). Fig. 3 tests the relationship for the total kinetic energy given by Eq. (4); the filtered kinetic energy is also shown, but this is known to be well predicted from earlier studies. All of these tests

Table 1
Scaling Relationships for the Full and Filtered Fields

	Full Field	Filtered Field
Dissipation	$\varepsilon \sim Q^3/L$	$\varepsilon_f \sim Q_f^3/L_f$
Energy Balance	$Q^2_{,t} = -2\varepsilon$	$Q^2_{f,t} = -2\varepsilon_f$
Kinetic Energy	$Q^2 \equiv \langle u_i u_i \rangle$	$Q_f^2 \equiv \langle \bar{u}_i \bar{u}_i \rangle$
Dissipation	$\varepsilon \equiv \langle 2\nu S_{ij} S_{ij} \rangle$	$\varepsilon_f \equiv \langle 2\nu_T \bar{S}_{ij} \bar{S}_{ij} \rangle$

validate the scaling laws for this flow and the constant in Eqs. (3) and (4) is found to be $c_1 = 1.04$.

All quantities were evaluated from the experimental data, except for ε_f. This rate of transfer has been calculated by LES using the vorticity model (see Kwak et al. (1975) and Mansour et al. (1978)). This model computes this rate of transfer well except for the first few time steps.

Similar tests show that these relationships work equally well for the other Comte-Bellot and Corrsin (1971) cases. There are insufficient data to do these tests for other flows at the present time.

III. Eddy-Viscosity Models

An eddy-viscosity model should accurately represent the effects of the subgrid-scale (or small-scale) motions on the resolvable flow field. There are two distinct components of the small scales in the decay of turbulence: a component that is correlated with the large scales and can be considered as resulting from energy transferred from the large scales, and a component that is the decaying remnant of the part of the initial small-scale field that is not correlated with the large-scale field..

In order to model the correlated component of the small scales, we shall invoke "production equals dissipation." We can then relate it to the current rate of energy transfer ε_f.

$$q_f^2 = c_2(2\varepsilon_f\Delta)^{2/3} \qquad (5)$$

This component will be analyzed in greater depth in Section IV, for now we note that $c_2 = .52$, which is exactly half of c_1.

An energy balance indicates that the energy in the uncorrelated component is, from Eqs. (3) and (5),

$$q_i^2 = c_1(2\varepsilon\Delta)^{2/3} - c_2(2\varepsilon_f\Delta)^{2/3} \qquad (6)$$

We expect that these two components will affect the large-scale motions differently and therefore propose the "two-component" model:

$$\nu_T = c_f q_f \Delta + c_i q_i \Delta \qquad (7)$$

This model can be reduced to one involving only q_f by use of the relations derived in Section II. In particular, q_i can be approximated by

$$q_i^2 = q_f^2 \; \frac{c_1}{c_2 - c_1\, q_f^2/Q_f^2} - 1 \qquad (8)$$

Finally, we need a way to compute q_f^2. This can be done from relationships already given and leads to a cubic equation for ν_T. We shall not give the result here, as a better approach will be given in Section IV.

For purposes of testing the concept, we also calculate the eddy viscosity directly by computing Q_f^2, q_f^2, and q_i^2 directly from the experimental data.

Simple Models

The two-component eddy-viscosity model differs from the Smagorinsky model. However, Kwak et al. (1975) and Shaanan et al. (1975) successfully simulated the (filtered) decay of turbulence using the Smagorinsky model, so we shall look for conditions that reduce the two-component model to the Smagorinsky model.

The correlated SGS component (q_f) is expected to interact more strongly with the large scales than does the uncorrelated component, so we expect that $c_f \gg c_i$. Then unless q_i is much larger than q_f, Eq. (7) reduces to a simple kinetic energy (TKE) model.

$$\nu_T = c_q q_f \Delta \qquad (9)$$

Moreover, if the filter width Δ is small, i.e., there are enough computational points such that $\Delta \ll L$, the uncorrelated component should be relatively small and can be neglected, and Eq. (7) again reduces to Eq. (9). This should also hold at long times in the decay of turbulence.

The Smagorinsky and vorticity models can be derived from the TKE model. In particular, using Eq. (5) and the last line in Table 1 and neglecting the spatial variation of ν_T, Eq. (9) can be reduced to

$$\nu_T = (c_s\Delta)^2 \langle 2\bar{S}_{ij}\bar{S}_{ij} \rangle^{1/2} \qquad (10)$$

If we assume that the vorticity scales like the strain rate, we obtain the vorticity model.

$$\nu_T = (c_v\Delta)^2 \langle \bar{\omega}_i\bar{\omega}_i \rangle^{1/2} \qquad (11)$$

Constants

Since we cannot simulate turbulent flows at high Reynolds numbers without modeling, the experiment of Comte-Bellot and Corrsin (1971) at $R_\lambda \approx 70$ was simulated with the vorticity model with $c_v = 0.213$, the value for which the filtered kinetic energy, $\frac{1}{2} Q_f^2$, is well represented (see Kwak et al., 1975, and Mansour et al., 1978). The computation was carried out on a 16^3 mesh using a pseudospectral program provided by Dr. Parviz Moin.

The excellent agreement between the filtered experimental decay and the computational results means that the constant in the eddy viscosity is accurate. From the eddy viscosity, the constant(s) of each model can be determined; these values are presented in Table 3. For the simple models the constants were evaluated from the simulated results at the last time step, $U_\infty t/M = 98$. For the two-component model, the constants were evaluated by a least-squares regression of the experimental and simulated results.

The constants of the simple models agree with previously reported values. Lilly (1967) derived the Smagorinsky model under the assumption that there is an inertial subrange, and found a slightly smaller constant $(c_s = 0.17)$, probably because the Kolmogorov spectrum overestimates the small-scale energy at low Reynolds numbers. In fact, Lilly's analysis gives

$$q_f^2 = 4.5\, \varepsilon_f^{2/3} \left(\frac{\Delta}{\pi}\right)^{2/3} \qquad (12)$$

Table 3
Model Constants

Model of the Eddy Viscosity		Constant
Two-component	$c_f q_f \Delta + c_i q_i \Delta$	$c_f = 0.107$
		$c_i = 0.011$
TKE	$c_q q_f \Delta$	$c_q = 0.126$
Smagorinsky	$(c_s \Delta)^2 < 2\bar{S}_{ij}\bar{S}_{ij} >^{1/2}$	$c_s = 0.197$
Vorticity	$(c_v \Delta)^2 < \bar{\omega}_i \bar{\omega}_i >^{1/2}$	$c_v = 0.213$

which overestimates q_f^2 by 25% for the Comte-Bellot and Corrsin (1971) flow. Moin et al. (1979) gave a model based on a Kolmogorov energy spectrum (inertial subrange) and a Gaussian filter. Their results overestimate the energy of the small scales of the Comte-Bellot and Corrsin (1971) experiment by 47%. If these differences are taken into account, the constants presented in Table 3 agree with these previous predictions.

These results apply only to high Reynolds number, which means $R_\lambda \gtrsim 40$. McMillan and Ferziger (1979) have given a low R_λ correction.

Deardorff (1970) and Schumann (1975) have simulated channel flow and found a Smagorinsky constant of 0.1 was required. The causes of this small constant are not well understood; nevertheless, we can offer a partial explanation.

- If the initial small-scale energy has been dissipated, the two-component model reduces to the TKE model with a smaller constant than that of the TKE model given in Table 3. The difference is not a 50% reduction of the Smagorinsky constant, but this explains part of the difference.

- There is evidence that the effect of mean shear on turbulence is to decrease the net rate of energy transfer to small scales (see McMillan, Ferziger, and Rogallo (1980)), and this could explain the smaller constant.

Decay of the Eddy Viscosity

Figure 4 presents the eddy-viscosity history for the decay of turbulence at $R_\lambda \approx 70$. The two-component and TKE models have been evaluated from experimental data, while the dissipation, Smagorinsky, and vorticity models have been evaluated using LES with the vorticity model as basic model. All values of the eddy viscosity are normalized with respect to the value at the last time step.

All models yield similar decay of the eddy viscosity because the 16^3 computational box is sufficient for $\Delta << L$, which is a condition for the validity of the simple models. The differences observed in the first time steps are mainly due to the decay of the uncorrelated SGS component, which is treated only by the two-component model.

Implications

The analysis of the eddy-viscosity models leads to the following conclusions:

- The eddy viscosity model should consider both components of the small-scale motions: the component corelated with the large scales, q_f, and the uncorrelated component, q_i. This leads to the two-component eddy-viscosity model.

- If $\Delta << L$, the TKE, dissipation, Smagorinsky, and vorticity models are reasonable approximations to the eddy viscosity in the simulation of the decay of turbulence.

- Previous workers assumed that the proper SGS velocity scale to be used in the model is q and that $q \sim |\bar{S}|\Delta$. In our view, both of these assumptions are incorrect but the resulting model is reasonable. One should use q_f as the velocity scale and $q_f \sim |\bar{S}|\Delta$ as a valid approximation.

IV. Filtered Small-Scale Motions

Since SGS models must be based on the resolved variables, it is reasonable to expect that the component of the small-scale motions which is related to the transfer of energy from the large scales, i.e., the correlated component, can be well-estimated from a knowledge of the large-scale motions. Moreover, we expect this component to be related to the small scales of the resolved flow field. A natural definition of the small-scale component of the resolved field is the difference between the filtered field (which is computed in LES) and the twice-filtered field (which can be obtained by filtering). Note that, since the small-scale field is given by

$$u' = u - \bar{u} \qquad (13)$$

we have, by filtering,

$$\overline{u'} = \bar{u} - \bar{\bar{u}} \qquad (14)$$

so that $\bar{u} - \bar{\bar{u}}$ is a reasonable estimate of the filtered SGS velocity, and the energy in the correlated SGS component can be estimated from

$$q_f^2 = < \bar{u}_i \bar{u}_i - \bar{\bar{u}}_i \bar{\bar{u}}_i > \qquad (15)$$

Then we can also evaluate the two-component and the TKE models from resolvable variables directly.

Test of the Models

The two-component, TKE, Smagorinsky, vorticity, and dissipation models were applied to the simulation of the decay of homogeneous isotropic turbulence at $R_\lambda \approx 70$. No significant differences are observed between the eddy viscosity models based on the filtered small-scale motions and the Smagorinsky, vorticity, and dissipation models. These results indicate that all these eddy-viscosity models model the effects of the small-scale motions on the large-scale variables in the decay of turbulence reasonably well.

Another Small-Length Scale

For flows in which the spectrum is different from the spectra obtained in the decay of homogeneous turbulence, it may be desirable to have a model which does not explicitly use the filter width Δ. Transition and relaminarizing flows may be examples of this. We shall develop such a model using the ideas given above. Cain (1980) suggested that a combination of the TKE and Smagorinsky models could provide this. Combining these models, we can obtain

$$\nu_T = 0.41 < \bar{u}_k\bar{u}_k - \bar{\bar{u}}_k\bar{\bar{u}}_k > < 2\bar{S}_{ij}\bar{S}_{ij} >^{-1/2} \quad (16)$$

This model is well behaved at a wall and should also behave well in a shear flow, according to Cain. This model is also dissipative.

A Model of the Reynolds Stresses

It may be possible to model the Reynolds stresses directly in terms of the small-scale component of the resolvable field. Such a model of the Reynolds stresses is

$$\tau_{ij} \sim \bar{u}_i\bar{u}_j - \bar{\bar{u}}_i\bar{\bar{u}}_j - \frac{1}{3}(\bar{u}_k\bar{u}_k - \bar{\bar{u}}_k\bar{\bar{u}}_k)\,\delta_{ij} \quad (17)$$

This model does not require the principal axes of the Reynolds stresses be aligned with the principal axes of the stress tensor S_{ij}.

Tests of this model by McMillan, Ferziger, and Rogallo indicate that this model correlates with the exact SGS Reynolds stresses better than the Smagorinsky model for decaying isotropic turbulence. More importantly, while the Smagorinsky model loses its validity in shear flows, this model is equally good in shear flow as in isotropic turbulence. It thus appears that this model is very promising. Note that it does not require a length scale.

However, the model has a serious flaw; it is not dissipative. We are therefore using a linear combination of this model and an eddy-viscosity model. A linear combination of this model and the one given in Section D may be even better.

Exact tests using data provided by McMillan, Ferziger, and Rogallo (1980) were performed for homogeneous isotropic turbulence at $R_\lambda = 38$ and for, one case of sheared homogeneous turbulence; see McMillan, Ferziger and Rogallo (1980) for further details. Tables 4 and 5 show the correlation coefficients between the exact SGS Reynolds stresses and the predictions of the Smagorinsky model, the model represented by Eq. (17), and a linear combination of both models. The combination is expected to have the best properties of both models.

Table 4

Average Correlation Coefficients in Homogeneous, Isotropic Turbulence at $R_\lambda = 38$ and $R_{SGS} = 180$

Model	Tensor Level	Vector Level	Scalar Level
Smagorinsky model	.24	.20	.30
New Model Eqn. (17)	.80	.71	.50
Combined model	.83	.74	.60

Table 5

Average Correlation Coefficients in Homogeneous Turbulence in the Presence of Mean Shear at $R_{SGS} = 208$

Model	Tensor Level	Vector Level	Scalar Level
Smagorinsky model	.05	.0	.05
New Model Eq. (17)	.80	.75	.58
Combined model	.80	.75	.58

A New Dissipative Model

The poor behavior of the Smagorinsky model in shear flows prompted us to search for other dissipative models with better properties. We took some guidance form the equations describing the SGS Reynolds stresses. In particular, one can expect the following model to be dissipative:

$$\tau_{ij} \sim c\Delta^2(\bar{u}_{i,k}\bar{u}_{j,k} - \frac{1}{3}\bar{u}_{\ell,k}\bar{u}_{\ell,k}\delta_{ij}) \quad (18)$$

Tests of the kind described above show that this model is only slightly inferior to the Smagorinsky model for isotropic turbulence but quite superior to it in sheared turbulence. We are therefore testing it as a candidate model. A combination of this model with Eq. (17) is also a strong candidate for a model with all of the desired properties.

V. Anisotropic Grids

Inertia Tensor Models

In the large eddy simulation of inhomogeneous shear flows, especially those in which there is a solid boundary, it is necessary to use filters and computational grids whose widths are different in each direction. We must expect that the eddy viscosity will no longer be a scalar; it can become a second- or fourth-rank tensor. Furthermore, it is not even clear which length scale should be used in an eddy viscosity model.

Most of the previous work with anisotropic meshes used the Smagorinsky model with a scalar eddy viscosity. The length scale in the model was based on the simplest scalar property of a parallelepiped--the volume:

$$\Delta = (\Delta_1\Delta_2\Delta_3)^{1/3} \quad (19)$$

This approach was first used by Deardorff (1970) and has been popular since.

To make a tensor eddy viscosity, we must find a tensor related to the geometry of a parallelepiped. *The simplest choice is the moment of inertia* tensor:

$$I_{ij} = \frac{1}{V}\int_V x_i x_j \, dV \qquad (20)$$

For a rectangular parallelepiped whose axes are aligned with the coordinate system, this becomes:

$$I_{ij} = \frac{2}{3}\begin{pmatrix} \Delta_1^2 & 0 & 0 \\ 0 & \Delta_2^2 & 0 \\ 0 & 0 & \Delta_3^2 \end{pmatrix} \qquad (21)$$

i.e., the tensor is on principal axes in this coordinate system.

This tensor can be decomposed into an isotropic component and a trace-free component that contains the anisotropy:

$$I_{ij} = I\delta_{ij} + (I_{ij} - I\delta_{ij}) = I\delta_{ij} + \hat{I}_{ij} \qquad (22)$$

where

$$I = \frac{1}{3} I_{kk} = \frac{1}{3}(\Delta_1^2 + \Delta_2^2 + \Delta_3^2) \qquad (23)$$

A model for τ_{ij} (which has no trace) can be constructed by creating the tensors that can be created from $I\delta_{ij}$, I_{ij}, $|\bar{S}|$, and $\bar{S}_{ij}$:

$$\tau_{ij} = c_1 I|\bar{S}|S_{ij} \qquad (24)$$
$$+ c_2|\bar{S}| \quad I_{ik}\bar{S}_{kj} + I_{jk}\bar{S}_{ki} - \frac{1}{3} I_{\ell k}\bar{S}_{k\ell}\delta_{ik}$$
$$+ c_3 \frac{|\bar{S}|}{I} \quad I_{ik}I_{k\ell}\bar{S}_{k\ell} - \frac{1}{3} I_{mk}I_{m\ell}\bar{S}_{k\ell}\delta_{ij}$$

We have chosen not to use tensor products involving $\bar{S}_{ij}$ more than once, but these could be added later if it appears that they are needed. We have also used only terms which reduce to the Smagorinsky model when the filter is isotropic. Note that we have used I_{ij} rather than $\hat{I}_{ij}$; the two are equivalent, and this choice simplifies the equations.

Scalar Eddy Viscosity

The first term in the Eq. (23) is just the usual Smagorinsky model with the length scale:

$$\Delta = \left(\frac{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}{3}\right)^{1/2} \qquad (25)$$

To test whether this length scale is superior to the one given by Eq. (19), we have used the data of McMillan and Ferziger (1978). They computed an "eddy viscosity" by dividing each component, τ_{ij}, by the corresponding component, $\bar{S}_{ij}$. This is not an eddy viscosity tensor, but it does provide an estimate of the relative importance of the isotropic and anisotropic parts of the eddy viscosity; there are no data available at present from which the eddy viscosity tensor can be constructed. The data do show, however, that the isotropic component of the eddy viscosity is much the most important, and they are sufficient to test whether Eq. (19) or Eq. (24) is a better choice. To do this, we have taken an average scalar eddy viscosity as the average of the values presented by McMillan and Ferziger (1978) and computed the "constant" that each of the two models would give.

The results are shown in Table 6 for several different anisotropic filters; the last line is the result for a case of strained homogeneous turbulence. It is clear from the table that, although it is not perfect, Eq. (24) is definitely superior to Eq. (19).

Much more *remains to be done in this area.* The anisotropic component of the eddy viscosity tensor needs to be computed (McMillan and Ferziger intend to do this) and compared with the models.

Table 6

Constants for the Smagorinsky Model for Anisotropic Grid Using Two Different Length Scales

Filter $(\Delta_1,\Delta_2,\Delta_3)$ (relative)	c_1, for Eq. (19)	c_1, for Eq. (24)
2,2,2	.152	.152
4,1,1	.292	.190
6,1,1	.352	.179
6,6,1	.297	.199
6,6,6	.182	.182
2.8,1.4,2	.160	.150

VI. Conclusions

1. The filtered flow field contains enough information to determine some of the characteristic scales of the full flow field, at least for the flows considered.

2. The small scale kinetic energy can be decomposed into two components, one correlated with the resolved field and one uncorrelated. While both components are significant parts of the total energy and dissipation, only the first component normally makes a significant contribution to the eddy viscosity.

3. The "production equals dissipation" argument does not hold in the decay of turbulence, because it disregards the decaying initial small scale motion (the uncorrelated component).

4. Previous authors have used the total SGS *energy* q^2 *and* $q \sim |\bar{S}|\Delta$ *as the basis for the* velocity scale in their SGS models. Neither of these assumptions is correct, but the compensating errors make the resulting model reasonable.

5. We have given a new eddy-viscosity model independent of the filter width (Eq. (16)) This model may be useful in transitional flows.

6. A new model of the Reynolds stresses which is not of the eddy-viscosity type (Eq. (17)) has been suggested and appears to be promising, especially when combined with an eddy viscosity model.

7. A new model which has better dissipating properties than the Smagorinsky model has been

suggested, and the combination of this model with the model of Eq. (17) appears to offer the potential for substantial improvement in large-eddy simulation techniques.

8. A new length scale for use in eddy viscosity models with anisotropic filters has been proposed and appears to be superior to the model which has been used heretofore.

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Appendix A

Filtering Process

An approach to LES that has been widely used to define the large-scale motions represented by the filtered flow field is computed, while the small-scale motions are modeled. This method is currently used by the Stanford and Queen Mary College groups and stated by Ferziger and Leslie (1979).

The filtering process is defined as:

$$\overline{\underline{u}}(\underline{x}) = \int \underline{u}(\underline{x}')\ G(\underline{x}',\underline{x})\ d\underline{x}'$$

where $\underline{u}$ is the velocity of the actual flow field, $\overline{\underline{u}}$ is the velocity of the filtered flow field, and G is the filter function, usually taken to be a Gaussian.

The Gaussian filter is defined as:

$$G(\underline{x}',\underline{x}) = \left(\sqrt{\frac{6}{\pi}}\ \frac{1}{\Delta}\right)^3 \exp\left[-6(\underline{x}-\underline{x}')^2/\Delta^2\right]$$

where Δ is the filter width. The filtered and unfiltered energy spectra are related by

$$E_f(k) = E(k)\ \exp\left[-k^2\Delta^2/12\right]$$

The Gaussian filter reduces the levels of the energy at smaller wave numbers (large-scale motions) and leaves almost no energy at the highest wave numbers (small-scale motions).

The energy spectrum of the actual flow field cannot be obtained from the energy spectrum of the filtered flow field because numerical approximations produce large errors at high wave numbers and no information is available at wave numbers above π/Δ.

Appendix B

Equations for Homogeneous Isotropic Turbulence

Complete flow field:

$$u_{i,i} = 0$$

$$u_{i,t} + u_j(u_{i,j} - u_{j,i}) = -\left(\frac{P}{\rho} + \frac{1}{2}u_ju_j\right)_{,i} + 2\nu S_{ij,j}$$

Filtered flow field:

$$\overline{u}_{i,i} = 0$$

$$\overline{u}_{i,t} + \overline{\overline{u}_j(\overline{u}_{i,j} - \overline{u}_{j,i})} = -\tilde{P}_{,i} - \tau_{ij} + 2\nu\overline{S}_{ij,j}$$

where

$$\tilde{P} = \frac{P}{\rho} + \frac{1}{2}\overline{\overline{u}_j\overline{u}_j} - \frac{1}{3}R_{kk}$$

$$\tau_{ij} = R_{ij} - \delta_{ij}R_{kk}$$

$$R_{ij} = \overline{u_i'u_j'} + \overline{u_i'\overline{u}_j} + \overline{\overline{u}_i u_j'} = +\overline{u_iu_j} - \overline{\overline{u}_i\overline{u}_j}$$

Eddy viscosity modeling:

$$\tau_{ij} = -2\nu_T\overline{S}_{ij}$$

The kinetic energy equation at high Reynolds number ($\nu_T >> \nu$) is:

$$\left\langle \frac{1}{2}\overline{u}_i\overline{u}_i \right\rangle_{,t} = -\left\langle 2\nu_T\overline{S}_{ij}\overline{S}_{ij}\right\rangle \equiv -\ \varepsilon_f$$

The dynamical system of equations of the filtered flow field (see Moin et al, 1978) can be solved once the model of the eddy viscosity (or Reynolds stresses) is formulated. The boundary conditions are periodic, and the initial conditions are formulated according to Kwak et al. (1975).

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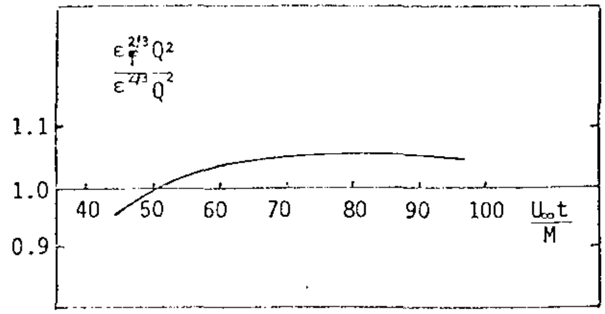

Fig. 1. Test of the dissipation scaling. The experimental data are those of Comte-Bellot and Corrsin (1971) with U_∞ = 10 m/sec and M = 0.0508 m. The filter used was a Gaussian, with Δ = 0.03 m, and ε_f was obtained by large-eddy simulation using the vorticity model with c_v = 0.2]3 (cf. Mansour et al. (1977)).

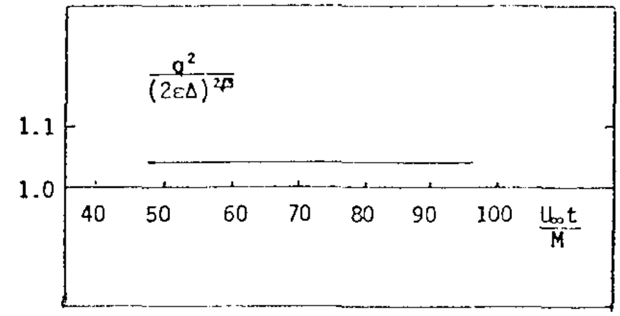

Fig. 2. Test of the scaling law for the small-scale energy given in Table 2. The parameters are the same as were used in Fig. 1.

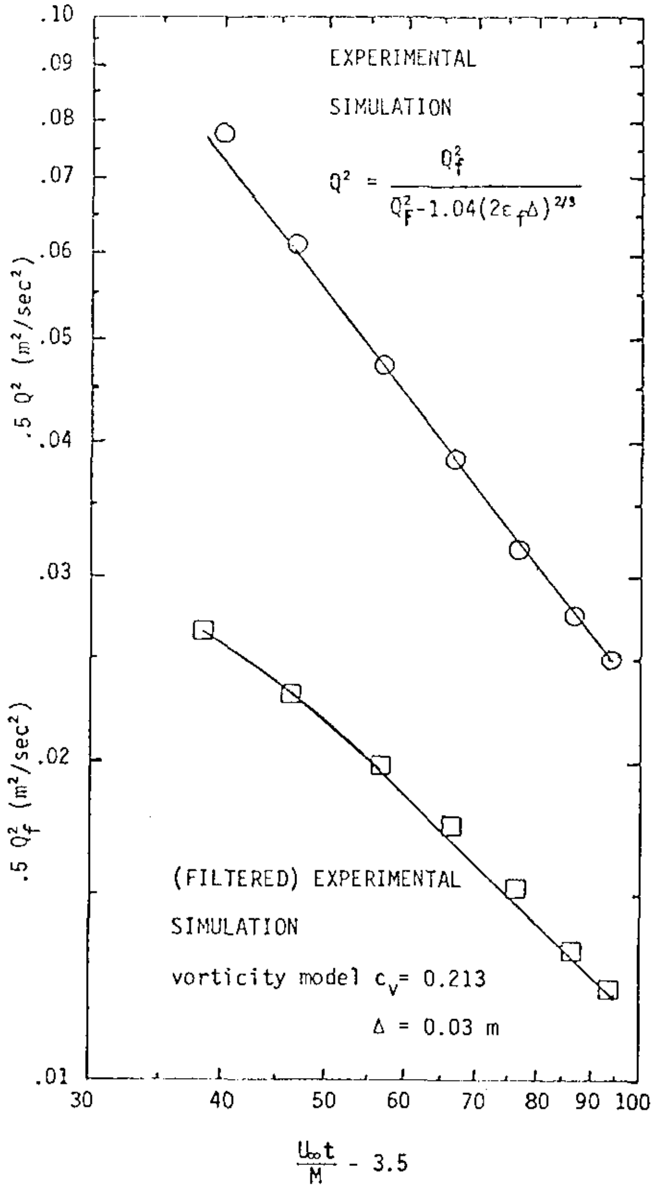

Fig. 3. Decay of turbulence for the flow of Comte-Bellot and Corrsin (1971) with U_∞ = 10 m/sec and M = 0.0508 m.

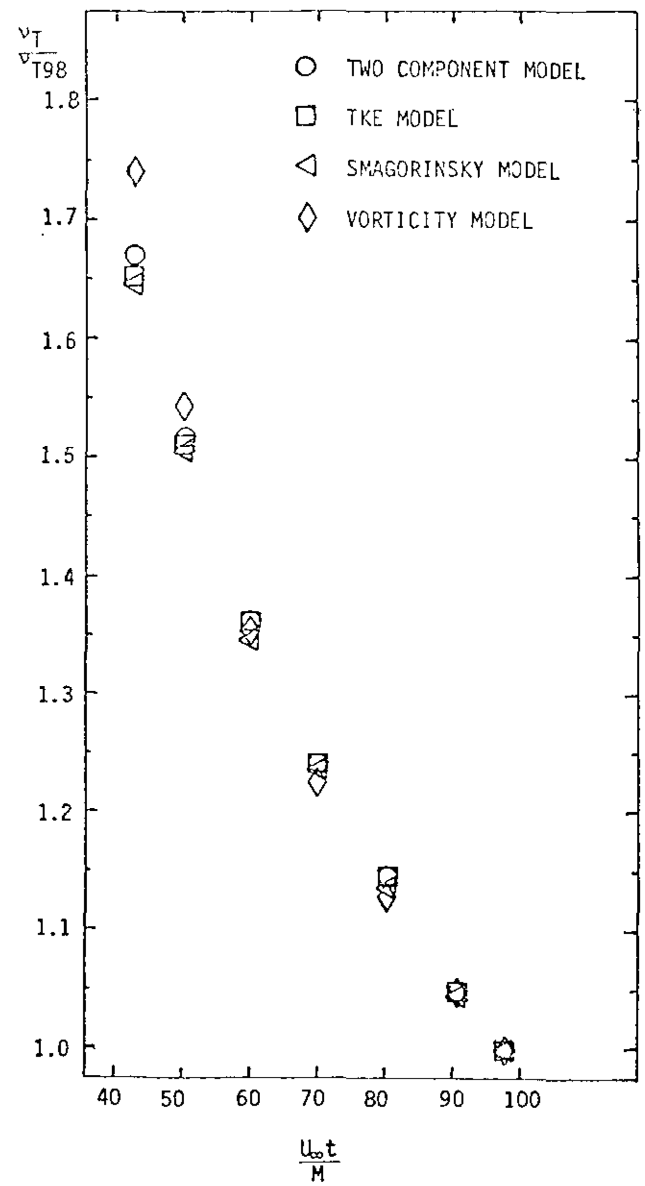

Fig. 4. Time history of the eddy viscosity for the flow of Comte-Bellot and Corrsin (1971) with U_∞ = 10 m/sec and M = 0.0508 m.