Environmental Fluid Dynamics: Lecture 21

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Overview

1. Monin-Obukhov Similarity Theory
    - Recap
    - Relationships
    - Surface-Layer Logarithmic Wind Profile
    - Aerodynamically Smooth and Rough Surfaces

MOST

Monin-Obukhov Similarity Theory Recap

Recall from the previous lecture that we used Buckingham Pi theory to relate non-dimensionalized gradients to fluxes in the atmospheric surface layer.

MOST Assumptions

- flow is quasi-stationary and horizontally-homogeneous
- turbulent fluxes are constant with height within the ASL.
- molecular exchanges are small compared to turbulent exchanges.
- rotational effects are neglected.
- influence of surface roughness, boundary-layer depth, and geostrophic wind are accounted for by τ_w/ρ.

Monin-Obukhov Similarity Theory Recap

Scaling variables

$$
\begin{aligned}
u_* &\simeq (-\overline{w'u'})^{1/2} \\
\theta_* &= -(\overline{w'\theta'})/u_* \\
\theta_{v*} &= -(\overline{w'\theta'_v})/u_* \\
q_* &= -(\overline{w'q'})/u_* \\
b_* &= -(\overline{w'b'})/u_* \\
L &= -u_*^3/(\kappa B_0) = u_*^2/\kappa b_*
\end{aligned}
$$

where L is the Obukhov Length, which describes the characteristic height of the sublayer of dynamic turbulence.

Monin-Obukhov Similarity Theory Recap

Similarity Theory

- Similarity Theory showed that mean flow variables or average turbulence quantities, when normalized by z, L, u_*, θ_*, etc., are functions of $\zeta = z/L$ only!
- ζ helps determine the relative importance of buoyancy versus shear effects, which makes it akin to the Richardson number (Ri).
  - $z \gg L$, buoyancy dominates
  - $z \ll L$, shear dominates

Monin-Obukhov Similarity Theory Recap

Similarity Theory

- We found these flux-profile relationships

$$\frac{\kappa z}{u_*}\frac{\partial \overline{u}}{\partial z} = \phi_m\left(\zeta\right) \qquad \frac{\kappa z}{\theta_*}\frac{\partial \overline{\theta}}{\partial z} = \phi_h\left(\zeta\right)$$

$$\frac{\kappa z}{\theta_{v*}}\frac{\partial \overline{\theta_v}}{\partial z} = \phi_v\left(\zeta\right) \qquad \frac{\kappa z}{b_*}\frac{\partial \overline{b}}{\partial z} = \phi_b\left(\zeta\right)$$

$$\frac{\kappa z}{q_*}\frac{\partial \overline{q}}{\partial z} = \phi_q\left(\zeta\right)$$

- Where ϕ terms are universal functions of z/L and we often assume $\phi_h = \phi_v = \phi_b = \phi_q$.

Monin-Obukhov Similarity Theory Recap

Similarity Theory

- We chose the empirical forms of the similarity functions as derived by Dyer (1974).

$$
\begin{array}{rll}
\text{neutral} & \phi_m = 1 & \phi_h = 1 \\
\text{unstable} & \phi_m = (1 - 16\zeta)^{-1/4} & \phi_h = (1 - 16\zeta)^{-1/2} \\
\text{stable} & \phi_m = 1 + 5\zeta & \phi_h = 1 + 5\zeta
\end{array}
$$

- In totality, MOST allows us to determine turbulent fluxes from the mean gradients (or gradients from fluxes)

Monin-Obukhov Similarity Theory Relationships

- Let's relate these functions to Ri
- The flux Richardson number and gradient Richardson number are, respectively:

$$\mathrm{Ri_f} = \frac{\overline{w'b'}}{\overline{w'u'}\partial\overline{u}/\partial z} \quad \text{and} \quad \mathrm{Ri} = \frac{\partial\overline{b}/\partial z}{(\partial\overline{u}/\partial z)^2}$$

- Recall our scales: $-\overline{w'u'} = u_*^2$ and $-\overline{w'b'} = u_* b_*$.
- And use our flux-profile relationships:

$$\frac{\kappa z}{u_*}\frac{\partial\overline{u}}{\partial z} = \phi_m \quad \text{and} \quad \frac{\kappa z}{b_*}\frac{\partial\overline{b}}{\partial z} = \phi_b$$

- With the Obukhov Length

$$L = -u_*^3/(\kappa B_0) = u_*^2/\kappa b_*$$

Monin-Obukhov Similarity Theory Relationships

- Flux Richardson number

$$\mathrm{Ri_f} = \frac{\overline{w'b'}}{\overline{w'u'}\partial\overline{u}/\partial z} = \frac{-u_* b_*}{-u_*^2 \partial\overline{u}/\partial z} = \frac{u_* b_* \kappa z}{u_*^3 \phi_m} = \frac{b_* \kappa z}{u_*^2 \phi_m} = \frac{z}{L\phi_m}$$

$$\boxed{\mathrm{Ri_f} = \zeta \phi_m^{-1}}$$

- Gradient Richardson number

$$\mathrm{Ri} = \frac{\partial\overline{b}/\partial z}{(\partial\overline{u}/\partial z)^2} = \frac{\dfrac{b_*}{\kappa z}\phi_h}{\dfrac{u_*^2}{(\kappa z)^2}\phi_m^2} = \frac{\kappa z b_* \phi_h}{u_*^2 \phi_m^2} = \frac{z\phi_h}{L\phi_m^2}$$

$$\boxed{\mathrm{Ri} = \zeta \frac{\phi_h}{\phi_m^2}}$$

Monin-Obukhov Similarity Theory Relationships

- Let's use K-theory to derive expressions that relate similarity functions to the turbulent Prandtl and Schmidt numbers.

$$-K_m \frac{\partial \overline{u}}{\partial z} = \overline{w'u'} \qquad -K_h \frac{\partial \overline{b}}{\partial z} = \overline{w'b'}$$

$$K_m \frac{\partial \overline{u}}{\partial z} = u_*^2 \qquad K_h \frac{\partial \overline{b}}{\partial z} = u_* b$$

$$K_m = \frac{u_*^2}{\frac{u_*}{\kappa z}\phi_m} \qquad K_h = \frac{u_* b_*}{\frac{b_*}{\kappa z}\phi_h}$$

$$\boxed{K_m = \frac{\kappa z u_*}{\phi_m}} \qquad \boxed{K_h = \frac{\kappa z u_*}{\phi_h}}$$

Monin-Obukhov Similarity Theory Relationships

- The Prandtl and Schmidt numbers are defined as:

$$\mathrm{Pr} = \nu/\nu_h \quad \text{and} \quad \mathrm{Sc} = \nu/\nu_q$$

- Analogously, we define their turbulent versions:

$$\mathrm{Pr_t} = K_m/K_h \quad \text{and} \quad \mathrm{Sc_t} = K_m/K_q$$

- Thus,

$$\mathrm{Pr_t} = \frac{\dfrac{\kappa z u_*}{\phi_m}}{\dfrac{\kappa z u_*}{\phi_h}} = \frac{\phi_h}{\phi_m} \qquad \mathrm{Sc_t} = \frac{\dfrac{\kappa z u_*}{\phi_m}}{\dfrac{\kappa z u_*}{\phi_q}} = \frac{\phi_q}{\phi_m}$$

- Recall, however, that we assume $\phi_q \approx \phi_h$, thus

$$\boxed{\mathrm{Pr_t} = \mathrm{Sc_t} = \frac{\phi_h}{\phi_m}}$$

Monin-Obukhov Similarity Theory Relationships

- Let's relate $\mathrm{Pr_t}$ and $\mathrm{Sc_t}$ to $\mathrm{Ri_f}$ and Ri :

$$\mathrm{Ri} = \zeta \frac{\phi_h}{\phi_m^2} = \mathrm{Ri_f} \frac{\phi_h}{\phi_m} = \mathrm{Ri_f}\mathrm{Pr_t} = \mathrm{Ri_f}\mathrm{Sc_t}$$

or

$$\boxed{\mathrm{Pr_t} = \mathrm{Sc_t} = \frac{\mathrm{Ri}}{\mathrm{Ri_f}}}$$

Monin-Obukhov Similarity Theory Relationships

- Consider *unstable* conditions using Dyer's functions

$$\phi_m = (1 - 16\zeta)^{-1/4} \qquad \phi_h = (1 - 16\zeta)^{-1/2}$$

$$\mathrm{Ri} = \zeta \frac{\phi_h}{\phi_m^2} = \zeta \leq 0$$
$$\mathrm{Ri_f} = \zeta \phi_m^{-1} = \zeta (1 - 16\zeta)^{1/4} \leq 0$$

Monin-Obukhov Similarity Theory Relationships

- Consider *stable* conditions using Dyer's functions

$$\phi_m = 1 + 5\zeta \qquad\qquad \phi_h = 1 + 5\zeta$$

$$\mathrm{Ri} = \zeta \frac{\phi_h}{\phi_m^2} = \zeta \phi_m^{-1} = \mathrm{Ri_f} = \frac{\zeta}{5\zeta + 1} \geq 0$$

- We can rearrange as

$$\zeta = \frac{\mathrm{Ri}}{1 - 5\mathrm{Ri}} \quad 0 \leq \mathrm{Ri} < 0.2$$

- Note that for $\mathrm{Ri} = 0.2$, $\zeta \to \infty$ $(L \to 0)$. This means that there is no turbulence beyond this value. Thus, the Dyer functions point to $\mathrm{Ri_c} = 0.2$.

Surface-Layer Logarithmic Wind Profile

- Consider the case of neutral stratification ($\phi_m = 1$)

$$\frac{\kappa z}{u_*}\frac{\partial \overline{u}}{\partial z} = 1$$

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{\kappa z} \quad \text{now integrate}$$

$$\boxed{u = \frac{u_*}{\kappa}\ln z + C}$$

  where C is a constant of integration.
- This describes the famous logarithmic wind profile in the atmospheric surface layer.
- Recall that wind should adhere to no-slip conditions ($u = 0$) at the surface. However, notice that there is discontinuity at $z = 0$. This points to the fact that the flow becomes laminar for very small z and brings about the concept of surface roughness.

Aerodynamically Smooth and Rough Surfaces

- If we take u_* as the velocity scale and δ_ℓ as the length scale of turbulence in the ASL, then the Reynolds number criterion for laminarization of the flow close to the surface (wall) is

$$\mathrm{Re}_\delta = \frac{u_* \delta_\ell}{\nu} \sim 1$$

  where ν is kinematic viscosity
- Thus, turbulence does not exist at distances from the wall of the order and less than $\delta_\ell \sim \nu / u_*$ (note: the oft-used viscous wall units are defined as $z^+ = z/\delta_\ell$ and $u^+ = \overline{u}/u_*$)
- Experimental data suggest that $\delta_\ell \sim 5\ \nu/u_*$, where the layer defined with this depth is called the *viscous sublayer*.

Aerodynamically Smooth and Rough Surfaces

Aerodynamically Smooth

- If roughness elements of characteristic size z_r are deployed in the viscous sublayer and $z_r \ll \delta_l$, then the surface is *aerodynamically smooth*.
- Lab data shows that surfaces are smooth for $z_r \leq 5\nu/u_*$.
- For the atmosphere, this corresponds to $z_r \lesssim 1 \text{ mm}$.
- However, most elements in the ASL are larger than 1 mm.
- Thus, most surfaces in the ASL are aerodynamically rough (exceptions: ice, mudflats, snow, water under light wind).

Aerodynamically Rough

- The surface is *aerodynamically rough* for $z_r \gg \delta_l$.
- Lab data shows that surfaces are rough for $z_r \geq 75\nu/u_*$.

Aerodynamic Surface Roughness Length

- In the case of a smooth surface, a turbulence flow regime represented by a logarithmic profile us possible at a height above ν/u_* (well above surface roughness elements).
- In the case of a rough surface, the flow is already turbulent in the near vicinity of surface roughness elements. Measurements show that $u = 0$ at some level close to z_r (actually just below).
- Let's introduce the idea of the surface roughness length.

Aerodynamic Surface Roughness Length

- Recall the generic log-law profile:

$$u = \frac{u_*}{\kappa} \ln z + C$$

- We will introduce a reference level z_0 where $u = 0$, defined through

$$C = -\left(\frac{u_*}{\kappa}\right) \ln z_0$$

- This leads to the neutral log-law profile

$$\boxed{u = \frac{u_*}{\kappa} \ln \frac{z}{z_0}}$$

where z_0 is called the aerodynamic surface roughness length (or surface roughness length) for momentum.

Mean Flow Above a Smooth Surface

- For smooth surfaces, z_0 defines the lower asymptotic limit of the logarithmic wind profile, below which the mean flow velocity is no longer a characteristic of turbulence.
- We can rearrange the neutral log-law expression and scale height by $\delta_\ell = \nu/u_*$

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{z}{\nu/u_*} + \frac{1}{\kappa} \ln \frac{\nu/u_*}{z_0}$$
$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{z}{\nu/u_*} + C_s$$

where

$$C_s = \frac{1}{\kappa} \ln \frac{\nu/u_*}{z_0}.$$

Lab data suggest that $C_s \approx 5$

Mean Flow Above a Smooth Surface

- The final form is given by

$$\boxed{\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{z u_*}{\nu} + 5} \quad \text{or} \quad \boxed{u^+ = \frac{1}{\kappa} \ln z^+ + 5}$$

- We can also approximate z_0:

$$C_s = \frac{1}{\kappa} \ln \frac{\nu / u_*}{z_0}$$

$$\frac{\nu / u_*}{z_0} = e^{\kappa C_s}$$

$$\boxed{z_0 = e^{-\kappa C_s} \frac{\nu}{u_*} \approx 0.1 \frac{\nu}{u_*}}$$

Or in other words, the surface roughness length for a smooth surface is approximately 10% of the viscous sublayer depth.

Mean Flow Above a Smooth Surface

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Or in other words, the surface roughness length for a smooth surface is approximately 10% of the viscous sublayer depth.

Mean Flow Above a Rough Surface

- For rough surfaces, z_0 is directly interpreted as the level where mean flow velocity vanishes. So,

$$u = \frac{u_*}{\kappa} \ln \frac{z}{z_0} \quad \text{where } u = 0 \text{ at } z = z_0$$

- In the real world, z_0 is a complex function of surface geometry, involving z_r as one of many parameters.
- Generally, z_0 increases with increasing z_r.

Mean Flow Above a Rough Surface

- In reality, there is no real consistent average velocity observed in a flow down to z_0 (below z_r).
- The velocity field obeys the log-law only at some distance $z \gg z_0$ above the surface.
- In this sense, z_0 is also the asymptotic limit of the logarithmic velocity profile.
- In order to make more applicable, we introduce the concept of the displacement height d.

$$u = \frac{u_*}{\kappa} \ln \frac{z-d}{z_0} \quad \text{where } u = 0 \text{ at } z = z_0 + d$$

- Far above the displaced height $(z \gg d)$, d is ignored

$$u = \frac{u_*}{\kappa} \ln \frac{z-d}{z_0} = u = \frac{u_*}{\kappa} \ln \frac{z/d-1}{z_0/d} \approx u = \frac{u_*}{\kappa} \ln \frac{z}{z_0}$$

$\mathbf{z_0}$ Parameterizations for Sand, Snow, Water

Snow, Sand

- z_0 for snow/sand increases with increasing wind speed.
- As wind speed increases, the material moves more actively and transports more effectively away from the surface.
- Empirical expression:

$$z_0 = \frac{\alpha_s u_*^2}{g}$$

where $\alpha_s = 0.016$ and $u_* > u_{*t}$. Here, $u_{*t} \approx 0.12 \text{ m s}^{-1}$ is a threshold frictions velocity. In the rough wall case, z_0 may be considered constant for snow/sand when $u_* < u_{*t}$.

$\mathbf{z_0}$ Parameterizations for Sand, Snow, Water

Water

- Wind generates waves on a water's surface.
- Waves occur within a broad range of geometric parameters (heights/lengths).
- Waves are generated and grow due to many physical mechanism, such as wave age, fetch, depth of the water body, and wind velocity (in terms of u_*).
- Roughness of wavy water is primarily determined by the steepest waves, rather than the longest.

$\mathbf{z_0}$ Parameterizations for Sand, Snow, Water

Water

- The shortest waves are capillary waves, with amplitudes/lengths $\mathcal{O}(1\ \mathrm{mm})$.
- Water is typically considered aerodynamically smooth if $\mathrm{Re}_* \ll 1$, so if we estimate $\mathrm{Re}_* = z_0 u_*/\nu \approx 0.1$, then $z_0 = m_s\left(\nu/u_*\right)$, where $m_s \approx 0.1$.
- Water is fully rough when $\mathrm{Re}_* \gg 1$. In this case we use $z_0 = \alpha_c u_*^2/g$, where α_c is the Charnock "constant", which ranges from $0.01 - 0.035$ (typically $0.014 - 0.019$).

$\mathbf{z_0}$ for Temperature and Moisture

- Boundary conditions for temperature and moisture at the underlying surface are formulated based on notions of their roughness lengths.

$$\theta = \theta_s \text{ at } z_{0\theta} \quad \text{and} \quad q = q_s \text{ at } z_{0q}$$

where $z_{0\theta}$ and z_{0q} are interpreted as the levels where θ and q reach their surface values θ_s and q_s, respectively.

$\mathbf{z_0}$ for Temperature and Moisture

- The physical nature of transport mechanisms for momentum, heat, and moisture differ significantly.
- e.g., pressure fluctuations are important to the transport of momentum, bu do not directly affect heat and moisture.
- Thus, there is no physical basis to expect that z_0 and $z_{0\theta}, z_{0q}$ should be the same, or even close.
- There are experimental indications of similarity between heat and moisture, so $z_{0\theta} \sim z_{0q}$.

Parameterizing the Relationships between $\mathbf{z_0}$ and $\mathbf{z_{0\theta}}, \mathbf{z_{0q}}$

- The number of roughness parameters needed is reduced by parameterizing relationships between them.
- Commonly, $z_0/z_{0\theta}$ and z_0/z_{0q} are parameterized based on the assumption that θ and q are logarithmic close to the surface.

$$\theta(z) = \theta_s + \frac{\theta_*}{\kappa} \ln \frac{z}{z_{0\theta}} \quad \text{and} \quad q(z) = q_s + \frac{q_*}{\kappa} \ln \frac{z}{z_{0q}}$$

thus,

$$\delta\theta = \theta(z_0) - \theta_s = \frac{\theta_*}{\kappa} \ln \frac{z}{z_{0\theta}} \quad \text{and} \quad \delta q = q(z_0) - q_s = \frac{q_*}{\kappa} \ln \frac{z}{z_{0q}}$$

- Experimental data suggest that $\ln(z_0/z_{0\theta})$ and $\ln(z_0/z_{0q})$ may be functions of $\mathrm{Re}_* = z_0 u_*/\nu$ for rough surfaces.

Parameterizing the Relationships between $\mathbf{z_0}$ and $\mathbf{z_{0\theta}, z_{0q}}$

- **Rough**

$$\frac{1}{\kappa}\ln\frac{z_0}{z_{0\theta}} = 6.2\ \mathrm{Re}_*{}^{1/4} - 5 \quad \text{and} \quad \frac{1}{\kappa}\ln\frac{z_0}{z_{0q}} = 5.7\ \mathrm{Re}_*{}^{1/4} - 5$$

- **Smooth**

$$\frac{1}{\kappa}\ln\frac{z_0}{z_{0\theta}} = 13.6\ \mathrm{Pr}^{2/3} - 12 \quad \text{and} \quad \frac{1}{\kappa}\ln\frac{z_0}{z_{0q}} = 13.6\ \mathrm{Sc}^{2/3} - 12$$

- Typical ASL values for Pr and Sc are 0.71 and 0.6, respectively, for smooth surfaces.
- Thus, $z_0/z_{0\theta} = 0.5$ and $z_0/z_{0q} = 0.3$ (i.e., the roughness lengths for heat and moisture are typically larger than that for momentum).