# Environmental Fluid Dynamics: Lecture 20

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# Overview

# Similarity Theory

# Similarity Theory

- **Goal:** We want to describe physical processes in the ABL.
- **Problem:** We lack understanding of the underlying physics.
- **Solution:** Derive empirical relationships b/t ABL variables.
- **Tool:** Similarity theory - group variables, create relationships.

# Similarity Theory

- Similarity theory is based on placing variables into dimensionless groups.
- We will use Buckingham Pi theory to do this.
- The goal is to properly group the variables such that we can create universal relationships between them.

# Similarity Theory

**Four Steps to Develop Similarity Theory**

- Choose relevant variables
- Organize variables into dimensionless groups
- Use experimental data to determine values of dimensionless groups
- Create bets-fit curve to describe the relationship between the variables

# Similarity Theory

- Result is an empirical equation or curves that show the same shape (i.e., they look self-similar - thus, similarity theory).
- We hope the result is universal so that we can apply it to other situations different to our experiment.
- The derived equations are called similarity relationships.
- These relationships are usually applied to steady-state situations.
- Think of similarity theory as a zero-order closure - we can use them to diagnose values of mean wind, temperature, and moisture as a function of height without making any assumptions regarding turbulence closure

# Buckingham Pi Theory

- Buckingham (1914) proposed a systematic approach for dimensional analysis.
- Buckingham Pi Theory represents an optimal approach to determine a dependent variable in a physical problem.
- If we can identify $m - 1$ parameters that govern a dependent variable, and if $n$ is the number of dimensions, then:
  - $m - n$ independent dimensionless quantities ($\pi$ groups) are formed (cannot be made from other $\pi$ groups)
  - $m - n$ independent dimensionless quantities are functionally related so that the dependent variable can be taken as a function of the governing parameters.
- The requires a grasp of a problem's physics.

**Buckingham Pi Theory - A procedure to group variables into dimensionless groups**

1. Select variables relevant to the problem
2. Find the dimensions of each variables and express in terms of fundamental dimensions: (e.g., length, mass, time, temp.)
3. Count the number of fundamental dimensions
4. Pick subset of the original variables as "key" variables, subject to these restrictions:
    - The number of key variables must be equal to the number of fundamental dimensions.
    - All fundamental dimensions must be represented in the key variables
    - No dimensionless group may be possible from any combination of the key variables

**Buckingham Pi Theory - A procedure to group variables into dimensionless groups**

5. Form dimensionless equations of the remaining variables in terms of the key variables
6. Solve for powers of the terms in the equations to yield dimensionally consistent equations
7. Divide the left hand side of each equation by the right to get dimensionless ($\pi$) group. The number of $\pi$ groups will always equal the number of variables minus the number of dimensions.

# Buckingham Pi Theory: Example

**Consider flow through a pipe. How does $\tau$ vary?**

1. We hypothesize that the important variables are fluid density, dynamic viscosity, velocity, shear stress, pipe diameter, and pipe roughness
2. The fundamental dimensions of these variables are:

| | | |
|---|---|---|
| fluid density | $\rho$ | $\mathrm{M\ L^{-3}}$ |
| dynamic viscosity | $\mu$ | $\mathrm{M\ L^{-1}\ T^{-1}}$ |
| velocity | $U$ | $\mathrm{L\ T^{-1}}$ |
| shear stress | $\tau$ | $\mathrm{M\ L^{-1}\ T^{-2}}$ |
| pipe diameter | $D$ | $\mathrm{L}$ |
| pipe roughness | $z_0$ | $\mathrm{L}$ |

# Buckingham Pi Theory: Example

**Consider flow through a pipe. How does $\tau$ vary?**

3. There are 3 fundamental dimensions: $\mathrm{M}, \mathrm{L}, \mathrm{T}$
4. We need 3 key variables. Let's choose $\rho$, $D$, and $U$.
5. Now we form dimensionless equations for $\mu$, $\tau$, and $z_0$ in terms of $\rho$, $D$, and $U$

$$\tau = \rho^a D^b U^c$$
$$\mu = \rho^d D^e U^f$$
$$z_0 = \rho^g D^h U^i$$

**Consider flow through a pipe. How does $\tau$ vary?**

6. Now we solve for the exponents. Let's look at $\tau$:

$$\tau = \rho^a D^b U^c$$
$$\mathrm{M\ L^{-1}\ T^{-2}} = (\mathrm{M\ L^{-3}})^a (L)^b (\mathrm{L\ T^{-1}})^c$$
$$\mathrm{M\ L^{-1}\ T^{-2}} = \mathrm{M}^a\ \mathrm{L}^{-3a+b+c}\ \mathrm{T}^{-c}$$

We must match dimensions

$$\mathrm{M} : 1 = a \qquad \mathrm{L} : -1 = -3a + b + c \qquad \mathrm{T} : -2 = -c$$

We solve for the unknowns to yield:

$$a = 1 \qquad b = 0 \qquad c = 2$$

# Buckingham Pi Theory: Example

**Consider flow through a pipe. How does $\tau$ vary?**

6. Thus, our dimensionally consistent equation is:

$$\tau = \rho^1 D^0 U^2 = \rho U^2$$

Similarly, we find that:

$$\mu = \rho U D \qquad z_0 = D$$

7. Now we divide the left by the right side to get our $\pi$ groups

$$\pi_1 = \frac{\tau}{\rho U^2} \qquad \pi_2 = \frac{\mu}{\rho U D} \qquad \pi_3 = \frac{z_0}{D}$$

Note that $\pi_1$ is the drag coefficient $C_D$, $\pi_2$ is inverse Reynolds number $\mathrm{Re}$, and $\pi_3$ is relative roughness.

# Scaling Variables

- For similarity theory, we want variables that represent forcings on the boundary layer (e.g., fluxes).
- Some key variables appear often and are called scaling variables.
- Generally, we want one length scale, one velocity scale, and if needed a temperature/moisture scale (usually no time scale since it can be made from length and velocity scales).
- Some variables always appear grouped, which allows for the creation of new scaling variables based on their combination.

# Scaling Variables

- Some common scaling variables for the atmosphere:

$$u_* = (-\overline{w'u'})^{1/2}$$

$$\theta_* = \frac{-(\overline{w'\theta'})}{u_*}$$

$$\theta_{v*} = \frac{-(\overline{w'\theta_v'})}{u_*}$$

$$q_* = \frac{-(\overline{w'q'})}{u_*}$$

$$b_* = \frac{-(\overline{w'b'})}{u_*}$$

# Scaling Variables

- Let's consider the signs of these scaling variables depending on static stability:

$$\text{unstable} \rightarrow \overline{w'b'} > 0, \partial b/\partial z < 0, b_* < 0$$
$$\text{neutral} \rightarrow \overline{w'b'} = 0, \partial b/\partial z = 0, b_* = 0$$
$$\text{stable} \rightarrow \overline{w'b'} < 0, \partial b/\partial z > 0, b_* > 0$$

- Notice how the scaling terms are aligned with the gradients.

# Monin-Obukhov Similarity Theory

- Theory developed for the atmosphere by Monin-Obukhov (1954) based on dimensional analysis.
- Monin-Obukhov Similarity Theory (MOST) suggests that there are four parameters governing quasi-steady-state turbulence immediately above a flat, horizontally-homogeneous surface

| | |
|---|---|
| $\ell = \kappa z$ | length scale of turbulence |
| $u_*$ | friction velocity |
| $B_0 = \overline{w'b'}$ | buoyancy flux |
| $\partial\overline{u}/\partial z$ | velocity gradient |

- Note: $\kappa$ is the von Kármán "constant", which is a dimensionless constant of proportionality introduced to relate the turbulence length scale and height above the surface. A typical value is $\kappa = 0.4$.

# Monin-Obukhov Similarity Theory

**It is also important to consider what we ignored:**

- *boundary layer depth*: assume largest eddies do not greatly influence eddies near the surface
- *mean wind*: turbulence must be invariant to Galilean transformations, and the mean wind is not
- *rotational effects*: turbulence Coriolis force is very small
- *molecular effects*: turbulence Reynolds number is very large
- *roughness elements* $z_0$: assume $z \gg z_0$

# Monin-Obukhov Similarity Theory

- The fundamental dimensions of our governing variables are:

| | | |
|---|---|---|
| turbulence length scale | $\ell$ | $\mathrm{L}$ |
| friction velocity | $u_*$ | $\mathrm{L}\ \mathrm{T}^{-1}$ |
| buoyancy flux | $B_0$ | $\mathrm{L}^2\ \mathrm{T}^{-3}$ |
| velocity gradient | $\partial\overline{u}/\partial z$ | $\mathrm{T}^{-1}$ |

- So we have $m = 4$ paramters and $n = 2$ dimensions.
- Accordingly, we expect to have $m - n = 2$ $\pi$ groups.
- We take group as a non-dimensionalized dependent variable and the other as the independent variable.

# Monin-Obukhov Similarity Theory

- Let's fidn the non-dimensionalized dependent variable
- There are 2 fundamental dimensions: $\mathrm{L}, \mathrm{T}$
- We need 2 key variables. Let's choose $u_*$ and $\kappa z$.
- Form a dimensionless equation for $\partial \overline{u}/\partial z$ in terms of $u_*$ and $\kappa z$:

$$\partial \overline{u}/\partial z = u_*^a \; (\kappa z)^b$$

- Now we solve for the exponents.

$$\partial \overline{u}/\partial z = u_*^a \; (\kappa z)^b$$

$$\mathrm{T}^-1 = (\mathrm{L} \; \mathrm{T}^{-1})^a \; (\mathrm{L})^b$$

$$\mathrm{T}^-1 = L^{a+b} \; T^{-a}$$

We must match dimensions

$$\mathrm{L} : 0 = a + b \qquad \mathrm{T} : -1 = -a$$

We solve for the unknowns to yield:

$$a = 1 \qquad b = -1$$

# Monin-Obukhov Similarity Theory

- Thus, our dimensionally consistent equation is:

$$\partial\overline{u}/\partial z = u_*^1\ (\kappa z)^{-1}$$

- Now we divide the left by the right side to get our $\pi$ groups

$$\pi_1 = \frac{\kappa z}{u_*}\frac{\partial\overline{u}}{\partial z}$$

- This represents the non-dimensional dependent variable (vertical gradient of velocity)

# Monin-Obukhov Similarity Theory

- Now, let's find the independent variable.
- There are 2 fundamental dimensions: $\mathrm{L}, \mathrm{T}$
- We need 2 key variables. Let's choose $u_*$ and $B_0$.
- Form a dimensionless equation for $\ell$ in terms of $u_*$ and $B_0$:

$$\ell = \kappa z = u_*^a B_0^b$$

- Now we solve for the exponents.

$$\begin{aligned} \kappa z &= u_*^a B_0^b \\ \mathrm{L} &= (\mathrm{L}\ \mathrm{T}^{-1})^a\ (\mathrm{L}^2\ \mathrm{T}^{-3})^b \\ \mathrm{L} &= L^{a+2b}\ T^{-a-3b} \end{aligned}$$

We must match dimensions

$$\mathrm{L} : 1 = a + 2b \qquad \mathrm{T} : 0 = -a - 3b$$

We solve for the unknowns to yield:

$$a = 3 \qquad b = -1$$

- Thus, our dimensionally consistent equation is:

$$\kappa z = u_*^3 \; B_0^{-1}$$

- Now we divide the left by the right side to get our $\pi$ groups

$$\pi_2 = \frac{\kappa z B_0}{u_*^3}$$

- Remember that we said some scaling variables always appear in a particular grouping? Here we define a new scaling variable called the Obukhov length,

$$L = -\frac{u_*^3}{\kappa B_0}$$

- Thus,

$$\pi_2 = -\frac{z}{L}$$

# Monin-Obukhov Similarity Theory

**Aside: Obukhov Length**

- $|L|$ is interpreted as the height at which buoyancy effects become dynamically important.
- Neutral conditions: $L \to \infty \Rightarrow z/L = 0$
- Stable conditions: $L > 0$
- Unstable conditions: $L < 0$
- In the absence of surface stress (no mean flow), $L = 0$

# Monin-Obukhov Similarity Theory

- The two $\pi$ groups are functionally related, thus

$$\frac{\kappa z}{u_*}\frac{\partial \overline{u}}{\partial z} = \phi_m\left(\frac{z}{L}\right)$$

where $\phi_m$ is a universal function of $\zeta = z/L$

- Similarly, we can show that

$$\frac{\kappa z}{\theta_*}\frac{\partial \overline{\theta}}{\partial z} = \phi_h\left(\zeta\right) \qquad \frac{\kappa z}{\theta_{v*}}\frac{\partial \overline{\theta_v}}{\partial z} = \phi_v\left(\zeta\right)$$

$$\frac{\kappa z}{b_*}\frac{\partial \overline{b}}{\partial z} = \phi_b\left(\zeta\right) \qquad \frac{\kappa z}{q_*}\frac{\partial \overline{q}}{\partial z} = \phi_q\left(\zeta\right)$$

Often we assume that $\phi_h = \phi_v = \phi_b = \phi_q$

- Thus, when normalized by $z, L, u_*, \theta_*, \theta_{v*}, b_*, q_*$, gradients of mean turbulent quantities are functions of only $\zeta = z/L$!

# Monin-Obukhov Similarity Theory

- We need formulations for our universal similarity functions
- Many empirical forms have been formulated using data from the famous 1968 Kansas experiment.

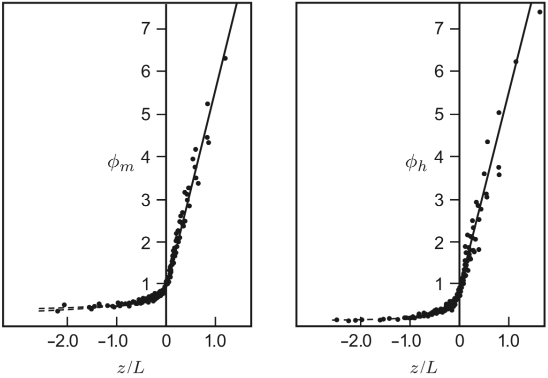


Figure 10.3 The M-O functions for mean wind shear (left) and mean potential temperature gradient (right), Eq. (10.12), from the 1968 Kansas experiment. From Businger *et al.* (1971).

From Wyngaard (2010)

# Monin-Obukhov Similarity Theory

- Although many forms exist, your professor prefers the functions proposed by Dyer (1974) because they are compact

| | | |
|---|---|---|
| neutral | $\phi_m = 1$ | $\phi_h = 1$ |
| unstable | $\phi_m = (1 - 16\zeta)^{-1/4}$ | $\phi_h = (1 - 16\zeta)^{-1/2}$ |
| stable | $\phi_m = 1 + 5\zeta$ | $\phi_h = 1 + 5\zeta$ |

- Thus, MOST allows us to determine turbulent fluxes from the mean gradients