Environmental Fluid Dynamics: Lecture 18

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Overview

1. Turbulence Kinetic Energy Balance

2. Buoyancy Variance Balance

Turbulence Kinetic Energy Balance

Turbulence Kinetic Energy Balance

- We previously derived the following generic expression for turbulent momentum flux.

$$
\begin{aligned}
\frac{\partial(\overline{u'_k u'_i})}{\partial t} = &-\overline{u}_j \frac{\partial(\overline{u'_k u'_i})}{\partial x_j} - \left[\overline{u'_j u'_i}\frac{\partial \overline{u}_k}{\partial x_j} + \overline{u'_k u'_j}\frac{\partial \overline{u}_i}{\partial x_j}\right] - \frac{\partial(\overline{u'_k u'_j u'_i})}{\partial x_j} \\
&+ \overline{u'_k b'}\delta_{i3} + \overline{u'_i b'}\delta_{k3} \\
&- \left[\frac{\partial(\overline{u'_k \Pi'})}{\partial x_i} + \frac{\partial(\overline{u'_i \Pi'})}{\partial x_k} - \overline{\Pi'\left(\frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k}\right)}\right] \\
&+ \nu \frac{\partial^2(\overline{u'_k u'_i})}{\partial x_j^2} - 2\nu \overline{\frac{\partial u'_k}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}
\end{aligned}
$$

- We will manipulate this equation to derive the turbulence kinetic energy (TKE) balance equation

Turbulence Kinetic Energy Balance

A pedantic aside

- You might often see TKE written as *turbulent* kinetic energy.
- No! Stop that!
- Writing turbulent kinetic energy implies that there is some kinetic energy that is itself turbulent.
- The more accurate name is *turbulence* kinetic energy.
- Here, we describe kinetic energy that arises due to eddies in a turbulent flow.

Turbulence Kinetic Energy Balance

- We want an expression for TKE ($\overline{e} = 0.5\overline{u_i'u_i'}$).
- So, we set $k = i$ and divide by 2:

$$\frac{1}{2}\frac{\partial(\overline{u_i'u_i'})}{\partial t} = -\frac{1}{2}\overline{u}_j\frac{\partial(\overline{u_i'u_i'})}{\partial x_j} - \frac{1}{2}\left[\overline{u_j'u_i'}\frac{\partial \overline{u}_i}{\partial x_j} + \overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j}\right] - \frac{1}{2}\frac{\partial(\overline{u_i'u_j'u_i'})}{\partial x_j}$$
$$+ \frac{1}{2}\overline{u_i'b'}\delta_{i3} + \frac{1}{2}\overline{u_i'b'}\delta_{k3}$$
$$- \frac{1}{2}\left[\frac{\partial(\overline{u_i'\Pi'})}{\partial x_i} + \frac{\partial(\overline{u_i'\Pi'})}{\partial x_i} - \overline{\Pi'\left(\frac{\partial u_i'}{\partial x_i} + \frac{\partial u_i'}{\partial x_i}\right)}\right]$$
$$+ \frac{1}{2}\nu\frac{\partial^2(\overline{u_i'u_i'})}{\partial x_j^2} - \frac{1}{2}2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

Turbulence Kinetic Energy Balance

- Substituting $\overline{e} = 0.5\overline{u_i' u_i'}$ and simplifying yields

$$\boxed{\begin{aligned}\underbrace{\frac{\partial \overline{e}}{\partial t}}_{1} = -\underbrace{\overline{u}_j \frac{\partial \overline{e}}{\partial x_j}}_{2} - \underbrace{\overline{u_j' u_i'} \frac{\partial \overline{u}_i}{\partial x_j}}_{3} - \underbrace{\frac{\partial (\overline{u_j' e})}{\partial x_j}}_{4} + \underbrace{\overline{u_i' b'} \delta_{i3}}_{5} \\ - \underbrace{\frac{\partial (\overline{u_i' \Pi'})}{\partial x_i}}_{6} + \underbrace{\nu \frac{\partial^2 \overline{e}}{\partial x_j^2}}_{7} - \underbrace{\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}}_{8}\end{aligned}} \quad (1)$$

Turbulence Kinetic Energy Balance

Terms in Eq. (1)

1. Storage of tke
2. Advection of tke by the mean wind
3. Production of tke by the mean wind shear
4. Transport of tke by turbulent motions (turbulent diffusion)
5. Production/destruction of tke by buoyancy
6. Transport of tke by pressure (pressure diffusion)
7. Molecular diffusion of tke
8. Viscous dissipation of tke

Turbulence Kinetic Energy Balance

Term 1

- $\partial\overline{e}/\partial t$ is very small at night, $\sim 2\ \mathrm{m}^2\ \mathrm{s}^{-2}$ during the day in the surface layer, and is often neglected over oceans.

Term 2

- Advection can vary widely, but is usually considered negligible over large areas $(10\ \mathrm{km} \times 10\ \mathrm{km})$.
- The term becomes important over smaller areas, where heterogeneity matters.

Turbulence Kinetic Energy Balance

Term 3

- Shear production describes the interaction between the flux and gradient.
- The effect is strongest at the surface.

Term 4

- Turbulent transport is advection by velocity fluctuations.
- This is not a source term because it integrates to zero over the entire domain (i.e., it redistributes TKE)

Term 5

- Buoyancy generally maxes at during the daytime hours at $\sim 0.25 \text{ K ms}^{-1}$
- Buoyancy is especially important during the day because it affects thermals.
- We can define the Deardorff scaling velocity for the mixed layer using vertical buoyancy flux.

$$w_* = (\overline{w'b'})^{1/3}$$

- For $b > 0$ (unstable), an air parcel displaced by turbulence continues to move in the direction of the displacement. Thus, there is production of
- For $b < 0$ (stable), an air parcel displaced by turbulence is forced to return to its starting point. Thus, there is destruction of TKE.

Turbulence Kinetic Energy Balance

Term 6

- Pressure fluctuations redistribute TKE.
- This is hard to measure and may be complicated by wave features.
- Accordingly, it is generally estimated as a residual.

Term 7

- Molecular diffusion ranges (units of $\mathrm{m}^2\ \mathrm{s}^{-3}$) from $\sim 10^{-11}$ in the ML to $\sim 10^{-7}$ in the SL
- The relatively small values mean that molecular effects are often neglected.

Turbulence Kinetic Energy Balance

Term 8

- The gradient $\partial u_i / \partial x_j$ is largest for small scales (i.e., shear is large for small-scale eddies).
- Viscous dissipation is always positive and ranges (units of $\mathrm{m}^2\ \mathrm{s}^{-3}$) from $\sim 10^{-4}$ in the ML to $\sim 10^{-2}$ in the SL
- The more intense the small-scale turbulence, the stronger the dissipation.
- Small-scale turbulence is driven by large-scale turbulence via the energy cascade.
- TKE and dissipation usually follow each other closely.

Vertical profiles of various TKE budget terms

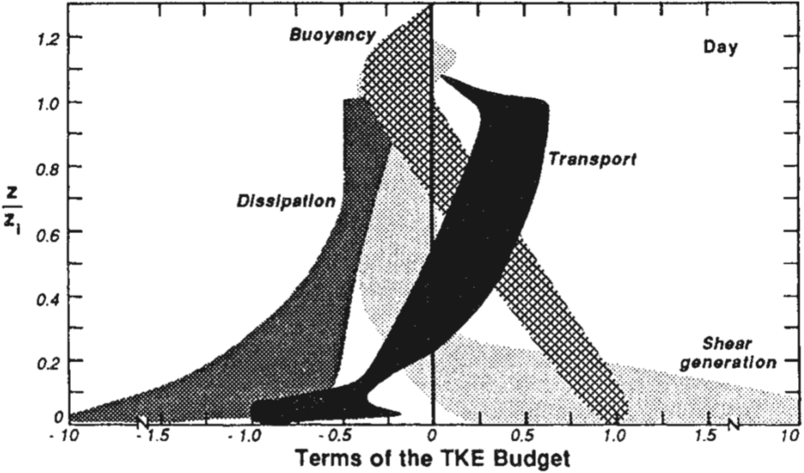

Fig. 5.4 from Stull (1988)

Buoyancy Variance Balance

Buoyancy Variance Balance

- We previously derived the following generic expression for turbulent buoyancy flux.

$$\frac{\partial b'}{\partial t} + \frac{\partial}{\partial x_j}\left[\overline{u_j} b' + u_j' \overline{b} + u_j' b' - \overline{u_j' b'}\right] = -N^2 u_j' \delta_{j3} + \nu_h \frac{\partial^2 b'}{\partial x_j^2}$$

- We will manipulate this equation to derive the buoyancy variance balance equation

Buoyancy Variance Balance

- We want an expression for buoyancy variance ($\overline{b'b'}$).
- So, we multiply by $2b'$ (2 so that we can invoke the product rule)

$$2b'\frac{\partial b'}{\partial t} + 2b'\frac{\partial}{\partial x_j}\left[\overline{u_j}b' + u_j'\overline{b} + u_j'b' - \overline{u_j'b'}\right] = -2b'N^2u_j'\delta_{j3} + 2b'\nu_h\frac{\partial^2 b'}{\partial x_j^2}$$

Buoyancy Variance Balance

- Make use of the product rule and then apply Reynolds averaging

$$\frac{\partial(\overline{b'b'})}{\partial t} + \overline{\overline{u_j}\frac{\partial(\overline{b'b'})}{\partial x_j}} + \overline{2u_j'b'\frac{\partial\bar{b}}{\partial x_j}} + \frac{\partial(\overline{u_j'b'b'})}{\partial x_j} - \overline{2b'\frac{\partial\overline{u_j'b'}}{\partial x_j}}$$
$$= -2N^2\overline{u_j'b'}\delta_{j3} + \nu_h\frac{\partial^2(\overline{b'b'})}{\partial x_j^2} - 2\nu_h\overline{\frac{\partial b'}{\partial x_j}\frac{\partial b'}{\partial x_j}}$$

- Simplification yields

$$\frac{\partial(\overline{b'b'})}{\partial t} + \overline{u_j}\frac{\partial(\overline{b'b'})}{\partial x_j} = -2\overline{u_j'b'}\frac{\partial\bar{b}}{\partial x_j} - \frac{\partial(\overline{u_j'b'b'})}{\partial x_j} - -2N^2\overline{u_j'b'}\delta_{j3}$$
$$+ \nu_h\frac{\partial^2(\overline{b'b'})}{\partial x_j^2} - 2\nu_h\overline{\frac{\partial b'}{\partial x_j}\frac{\partial b'}{\partial x_j}}$$

Buoyancy Variance Balance

- Grouping terms and simplifying yields

$$\boxed{\begin{aligned}\underbrace{\frac{\partial(\overline{b'b'})}{\partial t}}_{1} = &-\underbrace{\overline{u}_j \frac{\partial(\overline{b'b'})}{\partial x_j}}_{2} - \underbrace{2\overline{u'_j b'}\left[\frac{\partial \overline{b}}{\partial x_j} + N^2 \delta_{j3}\right]}_{3} \\ &- \underbrace{\frac{\partial(\overline{u'_j b' b'})}{\partial x_j}}_{4} + \underbrace{\nu_h \frac{\partial^2 (\overline{b'b'})}{\partial x_j^2}}_{5} - \underbrace{2\nu_h \overline{\frac{\partial b'}{\partial x_j}\frac{\partial b'}{\partial x_j}}}_{6}\end{aligned}} \tag{2}$$

Buoyancy Variance Balance

Terms in Eq. (2)

1. Storage of buoyancy variance
2. Advection of buoyancy variance by the mean wind
3. Production of buoyancy variance by the mean buoyancy shear + stratification
4. Transport of buoyancy variance by turbulence (turbulent diffusion)
5. Molecular diffusion of buoyancy variance
6. Viscous dissipation of buoyancy variance