Environmental Fluid Dynamics: Lecture 13

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Overview

1. Taylor-Proudman Theorem
2. Thermal Wind

Taylor-Proudman Theorem

Taylor-Proudman Theorem

- Consider the flow of a homogeneous flow that is in geostrophic balance.
- This flow is only observed in laboratory experiments because stratification effects cannot be avoided in nature.
- Imagine a tank with fluid that is steadily rotated at high angular speed Ω.
- At the same time, a solid body is moved slowly across the bottom of the tank.

Taylor-Proudman Theorem

- The angular speed Ω is made large, and the solid body is moved slowly, so that Coriolis $\gg$ acceleration terms.
- Acceleration terms must be negligible for geostrophic flow.
- Away from the frictional effects of the boundaries, the balance in this experiment is geostrophic in the horizontal and hydrostatic in the vertical.

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{3}$$

Taylor-Proudman Theorem

- Let's now define the Ekman number as the ratio of viscous to Coriolis forces (per unit volume):

$$E = \frac{\rho \nu U / L^2}{\rho f U} = \frac{\nu}{f L^2}$$

Based on the experimental setup, E is very small.

Taylor-Proudman Theorem

- First take $\partial/\partial y$ of Eq. (1):

$$-2\Omega\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial y}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial y}$$

- Next take $\partial/\partial x$ of Eq. (2):

$$2\Omega\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial y}$$

- Both equations are equal:

$$-2\Omega\frac{\partial v}{\partial y} = 2\Omega\frac{\partial u}{\partial x} \rightarrow 2\Omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- Recall that the incompressibility condition says $\vec{\nabla}\cdot\vec{U} = 0$. Therefore, $\partial w/\partial z = 0$.

Taylor-Proudman Theorem

- Next, differentiate Eqs. (1) and (2) with respect to z:

$$-2\Omega\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial z}$$
$$2\Omega\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial p}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial y}\frac{\partial p}{\partial z}$$

- Using Eq. (3):

$$-2\Omega\frac{\partial v}{\partial z} = \frac{\partial g}{\partial x} = 0 \qquad 2\Omega\frac{\partial u}{\partial z} = \frac{\partial g}{\partial y} = 0$$

- Both equations are equal:

$$2\Omega\frac{\partial v}{\partial z} = 2\Omega\frac{\partial u}{\partial z} \rightarrow \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

- We already showed that $\partial w/\partial z = 0$, so

$$\frac{\partial \vec{U}}{\partial z} = 0$$

Taylor-Proudman Theorem

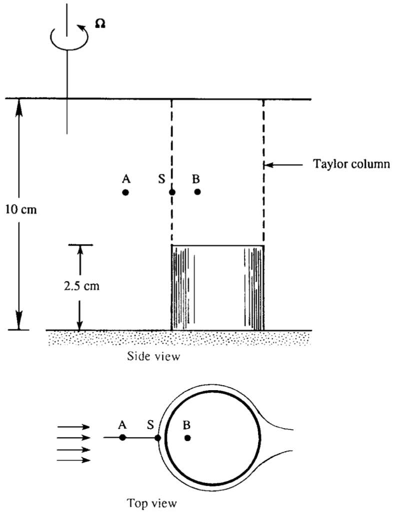
$$\frac{\partial \vec{U}}{\partial z} = 0$$

- This outcome shows that the velocity vector does not vary in the direction of the $\vec{\Omega}$.
- In other words, steady, slow motions in a rotating, inviscid, homogeneous fluid are two-dimensional.
- This is the **Taylor-Proudman theorem**.
- This theorem was derived by Proudman in 1916 and proved experimentally by Taylor soon thereafter.

Taylor-Proudman Theorem

Taylor's Experiment:

- Dye was released at point A, above the cylinder.
- If non-rotating, the dye would pass over the cylinder.
- If rotating, the dye split at point S, as if blocked by an extension of the cylinder, and flowed around this imaginary column.
- This was called a **Taylor column**.

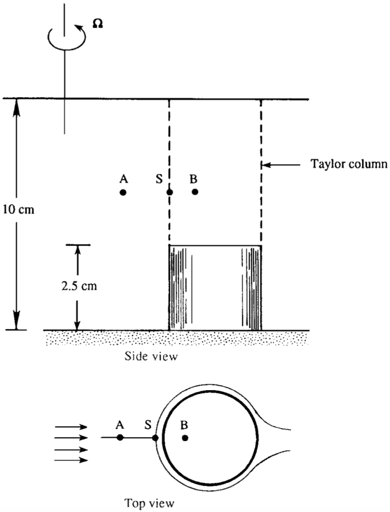


via: Kundu et al. (2008)

Taylor-Proudman Theorem

Taylor's Experiment:

- Dye released at point B moved with the cylinder.
- Conclusion: the flow outside of the vertical extension of the cylinder was the same as if the cylinder extended across the entire water depth.
- Conclusion: a column of water directly above the cylinder moved with it.



via: Kundu et al. (2008)

Taylor-Proudman Theorem

- For the case of a rotating steady, inviscid, homogeneous fluid, Taylor's experiments showed that bodies moving parallel or perpendicular to the axis of rotation carry with them a Taylor column of fluid.
- This Taylor column of fluid is oriented parallel to the axis of rotation.
- This phenomenon is similar to horizontal solid-body blocking in the real (stratified) world, such as flow encountering a mountain.

Thermal Wind

Thermal Wind

- Recall that the geostrophic wind is:

$$\overrightarrow{V_g} = \frac{1}{\rho f}\hat{k} \times \vec{\nabla} p$$

- We now define the **thermal wind** as:

$$\overrightarrow{V_T} = \overrightarrow{V_g}_{,\ \text{upper}} - \overrightarrow{V_g}_{,\ \text{lower}}$$

- The thermal wind is the vector difference between the geostrophic wind at some upper level and lower level.
- The name is a misnomer because it is not a wind.

Thermal Wind

- Why do we care about vertical changes in the geostrophic wind?
- Vertical changes in $\vec{V_g}$ (and hence the thermal wind) are associated with horizontal changes in temperature.
- Recall that the hydrostatic balance is given by:

$$\frac{\partial p}{\partial z} = -\rho g$$

we can apply the ideal gas law $p = \rho RT$ to get:

$$\frac{\partial p}{\partial z} = -\frac{pg}{RT}$$

Thermal Wind

- Consider an infinitesimally small difference in height δz between two adjacent pressure levels that are separated by the very small pressure difference δp:

$$\delta z = -\frac{RT}{pg}\delta p$$

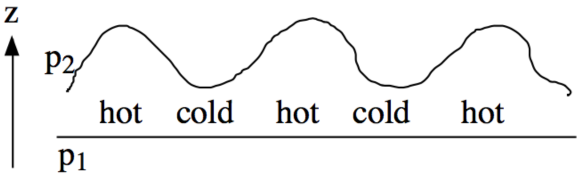
- Integrate to get the thickness between these two pressure levels spaced arbitrarily far apart:

$$z_2 - z_1 = -\frac{R}{g}\int_{p1}^{p_2} \frac{T}{p} dp$$

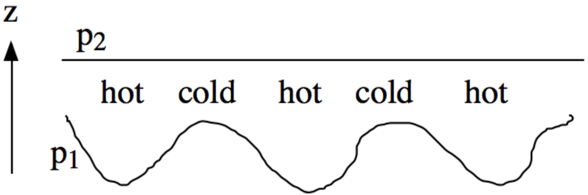
- Thus, the thickness of a layer is proportional to the temperature in the layer.

Thermal Wind

- As an example:



- Another example with the same temperature field:



- In both cases, the geostrophic wind changes with height because of horizontal temperature gradients.

Thermal Wind

- The expression for the thermal wind is messy!

$$\overrightarrow{V_T} = \overrightarrow{V}_{g,\text{ upper}} - \overrightarrow{V}_{g,\text{ lower}} = \frac{1}{\rho_{f\text{upper}}}\hat{k}\times\vec{\nabla}p_{\text{upper}} - \frac{1}{f\rho_{\text{upper}}}\hat{k}\times\vec{\nabla}p_{\text{lower}}$$

- We can make life easier if we switch to isobaric coordinates by using

$$\frac{1}{\rho}\vec{\nabla}_z p = \vec{\nabla}_p \Phi$$

where $\Phi = gz$. We get a much nicer expression:

$$\overrightarrow{V_T} = \frac{1}{f} \times \vec{\nabla}_p \left(\Phi_{\text{upper}} - \Phi_{\text{lower}}\right)$$

Thermal Wind

- Remember that the thermal wind is related to the vertical shear of the geostrophic wind:

$$\vec{V_g} = \frac{1}{f}\hat{k} \times \vec{\nabla}_p \Phi \qquad \text{take } \partial/\partial p$$

$$\frac{\partial \vec{V_g}}{\partial p} = \frac{1}{f}\hat{k} \times \vec{\nabla}_p \frac{\partial \Phi}{\partial p}$$

$$\frac{\partial p}{\partial z} = -\rho g \rightarrow [\div \text{ by } g \text{ and use } gz = \Phi] \rightarrow \frac{\partial p}{\partial \Phi} = -\rho$$

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} \rightarrow [\text{use ideal gas law}] \rightarrow \frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

We've now related $\partial\Phi/\partial p$ to T

Thermal Wind

- Continuing:

$$\frac{\partial \vec{V_g}}{\partial p} = \frac{1}{f}\hat{k} \times \vec{\nabla}_p \left(-\frac{RT}{p} \right) \quad p\text{=constant for isobaric level}$$

$$\frac{\partial \vec{V_g}}{\partial p} = -\frac{R}{fp}\hat{k} \times \vec{\nabla}_p T$$

$$\boxed{-\frac{\partial \vec{V_g}}{\partial p} = \frac{R}{fp}\hat{k} \times \vec{\nabla}_p T}$$

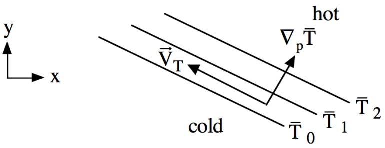
This is the **thermal wind relation**, although it is really an equation for the vertical shear of the geostrophic wind.

Thermal Wind

- Integrating the thermal wind relation will lead to the following general relationship in the Northern Hempisphere:

$$\overrightarrow{V_T} = (\text{positive values})\hat{k} \times \vec{\nabla}_p T$$

Thus, $\overrightarrow{V_T}$ is parallel to mean isotherms in a layer, with cold air to the left of $\overrightarrow{V_T}$.



- This describes the **Thermal Buys-Ballot Law**: "with $\overrightarrow{V_T}$ to your back, cold air is to your left."