Environmental Fluid Dynamics: Lecture 12

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Overview

1. The Ekman Layer
   - Overview
   - Boundary Conditions
   - Rotated Coordinate System
   - Navier-Stokes and Incompressibility
   - Solving the ODE
   - Profiles
   - Flow Characteristics
   - Ekman Pumping

The Ekman Layer

The Ekman Layer

Consider a scenario where fluid flow:

- is bounded below by a flat boundary,
- is purely geostrophic far above this boundary,
- is incompressible,
- is in a steady state,
- has constant density, and
- has constant Coriolis (latitude is taken as constant)

The Ekman Layer

- We want to derive an exact solution of the Navier-Stokes equations for this scenario.
- We anticipate that this flow behaves as:
  - $u(x, y, z, t) = u(z)$
  - $v(x, y, z, t) = v(z)$
  - $w(x, y, z, t) = 0$
- That is, the flow is in a steady state, has no vertical motion, and is horizontally-uniform in any horizontal plane.

The Ekman Layer: Boundary Conditions

Lower Boundary Conditions

- The ground $(z = 0)$ is impermeable $(w = 0)$, which is automatically satisfied since w is assumed to be zero everywhere.
- Since this is a viscous flow, we must impose the no-slip condition at the surface $[u(0) = 0, v(0) = 0]$.

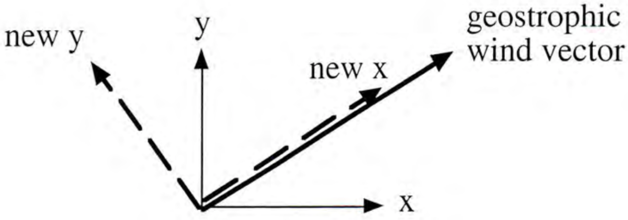
Upper Boundary Conditions

- The scenario says that the flow is purely geostrophic far away from the lower boundary $[u(\infty) = u_g,\ v(\infty) = v_g]$, where the geostrophic components are given by:

$$-fv_g = -\frac{1}{\rho}\frac{\partial p}{\partial x}(\infty) \qquad fu_g = -\frac{1}{\rho}\frac{\partial p}{\partial y}(\infty)$$

The Ekman Layer: Rotated Coordinate System

- We can make life easier if we consider a new Cartesian coordinate system where the x-axis is aligned in the direction of the geostrophic wind vector.
- Thus, the y-axis points int he direction of $-\vec{\nabla}p$ (toward low pressure).



The Ekman Layer: Rotated Coordinate System

- In this new coordinate system:

$$u_g(\infty) = U_g, \quad v_g(\infty) = 0, \quad \text{where} \quad U_g = \sqrt{u_g^2 + v_g^2}$$

- And the PGF at ∞ satisfies:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x}(\infty), \quad fU_g = -\frac{1}{\rho}\frac{\partial p}{\partial y}(\infty)$$

- The no-slip condition remains:

$$u(0) = 0, \quad v(0) = 0$$

The Ekman Layer: Navier-Stokes and Incompressibility

- Since $u = u(z)$, $v = v(z)$, and $w = 0$, the incompressibility condition ($\vec{\nabla} \cdot \vec{U} = 0$) is automatically satisfied.
- The Navier-Stokes equations reduce to:

$$0 = -\frac{1}{\rho}\vec{\nabla}p - 2\vec{\Omega} \times \vec{U} + \vec{g} + \nu\frac{\partial^2 \vec{U}}{\partial z^2}$$

or in component form:

x-component: $$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

y-component: $$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}$$

z-component: $$\frac{\partial p}{\partial z} = -\rho g$$

- So, hydrostatic balance in the vertical and a three-way balance between Coriolis, PGF, and friction in the horizontal.

The Ekman Layer: Navier-Stokes and Incompressibility

- Take x-derivative of the z-component equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = -\frac{\partial \rho}{\partial x} g = 0 \quad (\rho \text{ is constant})$$

  Interchanging the order of differentiation yields:

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = 0$$

- So the x-component PGF is independent of height!
- Similarly, the y-component PGF is also independent of height.

The Ekman Layer: Navier-Stokes and Incompressibility

- Accordingly, the horizontal PGF at any height is equal to the horizontal PGF at $z = \infty$:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}(\text{any } z) = -\frac{1}{\rho}\frac{\partial p}{\partial x}(\infty) = 0 \qquad \text{(via top bound. condition)}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}(\text{any } z) = -\frac{1}{\rho}\frac{\partial p}{\partial y}(\infty) = fU_g \qquad \text{(via top bound. condition)}$$

- We can use these expression to rewrite the x- and y-components of the Navier-Stokes equations

$$x\text{-component:} \qquad -fv = \nu\frac{\partial^2 u}{\partial z^2} \tag{1}$$

$$y\text{-component:} \qquad fu = fU_g + \nu\frac{\partial^2 v}{\partial z^2} \tag{2}$$

The Ekman Layer: Navier-Stokes and Incompressibility

- Eqs. (1) and (2) represent two equations in two unknowns (u and v).
- We want one equation and one unknown.
- Use Eq. (1) to express v in terms of u:

$$v = -\frac{\nu}{f}\frac{\partial^2 u}{\partial z^2}$$

- Now substitute this into Eq. (2):

$$fu = fU_g + \nu\frac{\partial^2}{\partial z^2}\left(-\frac{\nu}{f}\frac{\partial^2 u}{\partial z^2}\right)$$

This is one equation and in unknown (u).

The Ekman Layer: Navier-Stokes and Incompressibility

- If we multiply by f/ν^2 and rearrange, we arrive at a 4$^{\text{th}}$-order linear inhomogeneous constant-coefficient ordinary differential equation (ODE) for u:

$$\frac{\partial^4 u}{\partial z^4} + \frac{f^2}{\nu^2}(u - U_g) = 0$$

- Since U_g is a constant, we can subtract it from u in the first term. This will make the ODE homogeneous.

$$\frac{\partial^4}{\partial z^4}(u - U_g) + \frac{f^2}{\nu^2}(u - U_g) = 0$$

- Finally, we define a new independent variable $\tilde{u} = u - U_g$, where $\tilde{u}$ is the x-component of the ageostrophic wind. The ODE is now linear, constant-coefficient, and homogeneous.

$$\boxed{\frac{\partial^4 \tilde{u}}{\partial z^4} + \frac{f^2}{\nu^2}\tilde{u} = 0} \qquad (3)$$

The Ekman Layer: Solving the ODE

- To solve the ODE, we seeks solutions of the form: $\tilde{u} = e^{mz}$.
- Plugging this into Eq. (3) yields:

$$\begin{aligned} \left(m^4 + f^2/\nu^2\right) e^{mz} &= 0 \\ m^4 + f^2/\nu^2 &= 0 \\ m^4 &= -f^2/\nu^2 && \text{take root} \\ m^2 &= \pm i f/\nu && \text{take root again} \\ m &= \pm\sqrt{\pm 1}\sqrt{i}\sqrt{f/\nu} \end{aligned}$$

$\sqrt{\pm 1} = 1$ or i, but what is $\sqrt{i}$?

The Ekman Layer: Solving the ODE

- Recall Euler's formula:

$$\boxed{e^{im} = \cos m + i \sin m} \tag{4}$$

Thus,

$$\begin{aligned} e^{im+2\pi ni} &= e^{i(m+2\pi n)} = \cos(m+2\pi n) + i\sin(m+2\pi n) \\ &= \cos m + i \sin q \quad \text{(assuming } n \text{ is an integer)} \\ &= e^{im} \end{aligned}$$

$$\boxed{e^{im+2\pi ni} = e^{im}} \tag{5}$$

The Ekman Layer: Solving the ODE

- Setting $m = \pi/2$ in Euler's formula, Eq. (4), yields:

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = 0 + 1i = i$$

- Using Eq. 5:

$$i = e^{i\pi/2} = e^{i\pi/2 + 2\pi n i} \quad (n \text{ is an integer})$$

- Now we take the root of i:

$$\begin{aligned}\sqrt{i} = i^{1/2} &= e^{i\pi/4 + \pi n i} \\ &= \cos(\pi/4 + \pi n) + i\sin(\pi/4 + \pi n)\end{aligned}$$

- Let's evaluate this expression for various value of n.

The Ekman Layer: Solving the ODE

- $n = 0$:

$$\begin{aligned} i^{1/2} &= \cos(\pi/4) + i\sin(\pi/4) \\ &= \frac{1}{\sqrt{2}}(1+i) \end{aligned}$$

- $n = 1$:

$$\begin{aligned} i^{1/2} &= \cos(\pi/4+\pi) + i\sin(\pi/4+\pi) \\ &= -\frac{1}{\sqrt{2}}(1+i) \end{aligned}$$

- You can show that $n = 2$ is the same as for $n = 0$.
- You can show that $n = 3$ is the same as for $n = 1$.
- Thus, $\sqrt{i}$ has two distinct roots.

- Thus, there are four possible solutions for $m = \pm\sqrt{\pm 1}\sqrt{i}\sqrt{f/\nu}$:

$$m_1 = \frac{1}{\sqrt{2}}(1+i)\sqrt{\frac{f}{\nu}}$$

$$m_2 = \frac{1}{\sqrt{2}}(-1-i)\sqrt{\frac{f}{\nu}}$$

$$m_3 = im_1 = \frac{1}{\sqrt{2}}(-1+i)\sqrt{\frac{f}{\nu}}$$

$$m_4 = im_2 = \frac{1}{\sqrt{2}}(1-i)\sqrt{\frac{f}{\nu}}$$

The Ekman Layer: Solving the ODE

- Let's define the **Ekman Depth** as:

$$\delta_E \equiv \sqrt{\frac{2\nu}{f}}$$

- Then we can rewrite the four roots of m as:

$$m_1 = \frac{1+i}{\delta_E}$$
$$m_2 = \frac{-1-i}{\delta_E}$$
$$m_3 = \frac{-1+i}{\delta_E}$$
$$m_4 = \frac{1-i}{\delta_E}$$

The Ekman Layer: Solving the ODE

- Using our assumed form, the general solution for $\tilde{u}$ is:

$$\tilde{u} = ae^{m_1 z} + be^{m_2 z} + ce^{m_3 z} + de^{m_4 z}$$

  where $a, b, c,$ and d are constants.

- Substitute our expression for $m_1, m_2, m_3,$ and m_4:

$$\tilde{u} = ae^{(1+i)z/\delta_E} + be^{(-1-i)z/\delta_E} + ce^{(-1+i)z/\delta_E} + de^{(1-i)z/\delta_E}$$

- We must apply our boundary conditions to solve for the constants.

The Ekman Layer: Solving the ODE

- Start with the upper boundary condition.
- Recall $u(\infty) = U_g$ and $\tilde{u} = u - U_G$, thus
  $\tilde{u}(\infty) = u(\infty) - U_g = 0$

$$0 = \lim_{z \to \infty} \left[ae^{(1+i)z/\delta_E} + be^{(-1-i)z/\delta_E} + ce^{(-1+i)z/\delta_E} + de^{(1-i)z/\delta_E} \right]$$

- Look at the real part of these exponentials as $z \to \infty$:
  - e^z/δ_E blows up
  - e^-z/δ_E goes to zero
- We must set $a = d = 0$ to prevent the solutions from blowing up. This leaves:

$$\tilde{u} = be^{(-1-i)z/\delta_E} + ce^{(-1+i)z/\delta_E} \quad (6)$$

The Ekman Layer: Solving the ODE

- To solve for b and c, let's apply the lower no-slip boundary condition.
- One is $u(0) = 0$. Recall $\tilde{u} = u - U_g$, so $\tilde{u}(0) = -U_g$
- Applying this to Eq. (6):

$$-U_g = b + c$$

- The other no-slip condition is $v(0) = 0$. We want an expression for v (valid everywhere) and then evaluate it at $z = 0$.
- Recall that the x-component Navier-Stokes equation gave:

$$v = -\frac{\nu}{f}\frac{\partial^2 u}{\partial z^2} = -\frac{\nu}{f}\frac{\partial^2 \tilde{u}}{\partial z^2}$$

The Ekman Layer: Solving the ODE

- The first derivative of $\tilde{u} = be^{(-1-i)z/\delta_E} + ce^{(-1+i)z/\delta_E}$ is:

$$\frac{\partial \tilde{u}}{\partial z} = b\frac{(-1-i)}{\delta_E}e^{(-1-i)z/\delta_E} + c\frac{(-1+i)}{\delta_E}e^{(-1+i)z/\delta_E}$$

- Taking the second derivative yields:

$$\frac{\partial^2 \tilde{u}}{\partial z^2} = b\frac{(-1-i)^2}{\delta_E^2}e^{(-1-i)z/\delta_E} + c\frac{(-1+i)^2}{\delta_E^2}e^{(-1+i)z/\delta_E}$$

$$(-1-i)^2 = (1+i)(1+i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$$
$$(-1+i)^2 = (-1+i)(-1+i) = 1 - 2i + i^2 = 1 - 2i - 1 = -2i$$

The Ekman Layer: Solving the ODE

- Substitution gives us:

$$\begin{aligned} v &= -\frac{\nu}{f}\frac{\partial^2 \tilde{u}}{\partial z^2} \\ &= -\frac{2i\nu}{f\delta_E^2}\left[be^{(-1-i)z/\delta_E} - ce^{(-1+i)z/\delta_E}\right] \end{aligned}$$

- Using the definition of the Ekman Depth, $\delta_E^2 = 2\nu/f$:

$$v = -i\left[be^{(-1-i)z/\delta_E} - ce^{(-1+i)z/\delta_E}\right] \quad (7)$$

- Applying the no-slip condition $v(0) = 0$ to Eq (7):

$$0 - -i(b - c) \rightarrow b = c$$

- Combining with our previous result, $-U_g = b + c$:

$$b = c = -U_g/2$$

The Ekman Layer: Solving the ODE

- Apply our values of b and c to Eq. (6):

$$\tilde{u} = -\frac{U_g}{2}\left[e^{(-1-i)z/\delta_E} + e^{(-1+i)z/\delta_E}\right]$$

- Factor out the real exponential:

$$\tilde{u} = -\frac{U_g}{2}e^{-z/\delta_E}\left[e^{-iz/\delta_E} + eiz/\delta_E\right]$$

- Now we can expand the complex exponentials using Euler's formula:

$$\tilde{u} = -\frac{U_g}{2}e^{-z/\delta_E}\left[\cos(-z/\delta_E) + i\sin(-z/\delta_E)\right.$$
$$\left. + \cos(z/\delta_E) + i\sin(z/\delta_E)\right]$$

The Ekman Layer: Solving the ODE

- Note that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$:

$$\tilde{u} = -U_g e^{-z/\delta_E} \cos(z/\delta_E)$$

- Finally, since $\tilde{u} = u - U_g$, we obtain $u = \tilde{u} + U_g$:

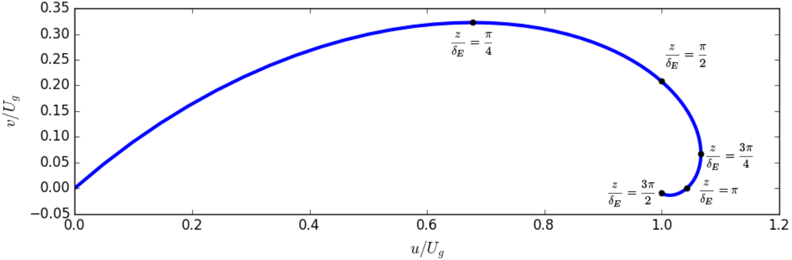
$$\boxed{u = U_g \left[1 - e^{-z/\delta_E} \cos(z/\delta_E)\right]} \tag{8}$$

- Similarly, we can evaluate Eq. (7) to obtain:

$$\boxed{v = U_g e^{-z/\delta_E} \sin(z/\delta_E)} \tag{9}$$

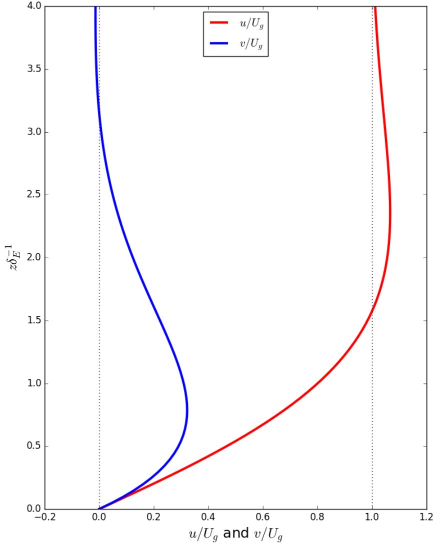
The Ekman Layer: Hodograph

The classic Ekman spiral



The Ekman Layer: Vertical Profiles

Vertical profiles



The Ekman Layer: Ekman Flow

- From the Ekman solution we see that friction induces a flow component directed toward low pressure.
- Ekman Depth δ_E is the measure of frictional boundary layer thickness.
- At $z = \delta_E$, the wind is approximately 80% geostrophic.
- $\delta_E = \sqrt{2\nu/f}$
  - As friction increases, the thickness increases
  - As Coriolis increases, the thickness decreases

The Ekman Layer: Ekman Flow

- The observed Ekman depths in the atmosphere are on the order of 1000 m.
- Theory says:

$$\delta_E = \sqrt{\frac{2\nu}{f}} = \sqrt{\frac{2 \times 1.4 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}}{10^{-4} \text{ s}^{-1}}} \sim -0.5 \text{ m}$$

- Those ... are ... not close! Why?
- The atmosphere is turbulent, so $\vec{U} = \vec{U}(x, y, z, t)$ and not $\vec{U} = \vec{U}(z)$.
- However, if we take the spatial average of the Navier Stokes equations, the averaged equations look like the un-averaged equations but with molecular viscosity ν replaced with a much larger eddy-viscosity ν_E.

The Ekman Layer: Ekman Flow

- We can compute the eddy-viscosity based on the observed Ekman depth:

$$\sqrt{\frac{2\nu_E}{f}} = 1000 \text{ m} \rightarrow \nu_E = \frac{f}{2}(1000 \text{ m})^2 \sim 50 \text{ m}^2 \text{ s}^{-1}$$

- True Ekman spirals do not exist in nature.
- However, modified (flatter) spirals are observed, as well as the theoretical result that low-level flow cuts across isobars toward low-pressure.
- If streamlines are curved, Ekman theory is not strictly valid because u and v vary in x and y, respectively, as well as in z (but it's approximately valid).
- We will apply Ekman concepts locally by assuming that the velocity profile at a local point behaves like an Ekman velocity profile.

The Ekman Layer: Ekman Pumping

- The horizontal pressure gradient aloft is largely present at low levels.
- At low levels, friction induces a flow component toward low pressure.
- As a result, we get horizontal convergence into the low-pressure zone.
- This results in rising motion (from mass conservation).
- This can lead to condensation, rain, clouds, storms, etc.