Environmental Fluid Dynamics: Lecture 10

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Overview

1. Atmospheric Dynamics: Basic Equations
   - Conservation of Momentum, continued

Atmospheric Dynamics:

Conservation of Momentum,

continued

Conservation of Momentum: Rotating Coordinate System

- Apparent forces (centrifugal, Coriolis) will automatically appear when we write Newton's 2$^{\text{nd}}$ Law in a form appropriate for a rotating reference frame.
- Consider a Cartesian coordinate system x', y', and z' rotating with angular velocity $\vec{\Omega}$.
- Let $\hat{i}'$, $\hat{j}'$, $\hat{k}'$ be unit vectors in the x', y', and z' directions, respectively (these vectors rotate with the coordinate system).

Conservation of Momentum: Rotating Coordinate System

- Consider some arbitrary vector $\overrightarrow{A}$ that is observed in an inertial reference frame.
- Even though $\overrightarrow{A}$ is observed in a non-rotating reference frame, it can be decomposed into components in a rotating coordinate system:

$$\overrightarrow{A} = A'_x \hat{i}' + A'_y \hat{j}' + A'_z \hat{k}'$$

where

$$A'_x = \hat{i}' \cdot \overrightarrow{A}$$
$$A'_y = \hat{j}' \cdot \overrightarrow{A}$$
$$A'_z = \hat{k}' \cdot \overrightarrow{A}$$

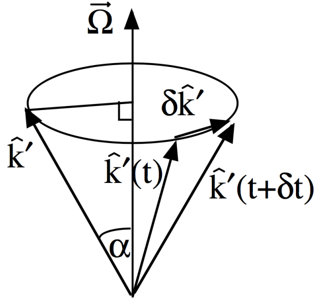
Conservation of Momentum: Rotating Coordinate System

- The total derivative of $\vec{A}$ as observed in the inertial reference frame is (where subscript a means absolute, or inertial):

$$
\begin{aligned}
\frac{D_a \vec{A}}{Dt} &= \frac{D_a}{Dt}\left(A'_x \hat{i}' + A'_y \hat{j}' + A'_z \hat{k}'\right) \\
&= \underbrace{\hat{i}' \frac{D_a A'_x}{Dt} + \hat{j}' \frac{D_a A'_y}{Dt} + \hat{k}' \frac{D_a A'_z}{Dt}}_{D\vec{A}/Dt \text{ observed in rotating ref frame}} \\
&\quad + A'_x \frac{D_a \hat{i}'}{Dt} + A'_y \frac{D_a \hat{j}'}{Dt} + A'_z \frac{D_a \hat{k}'}{Dt} \\
&= \frac{D\vec{A}}{Dt} + \underbrace{A'_x \frac{D_a \hat{i}'}{Dt} + A'_y \frac{D_a \hat{j}'}{Dt} + A'_z \frac{D_a \hat{k}'}{Dt}}_{\text{What are these terms?}}
\end{aligned}
$$

Conservation of Momentum: Rotating Coordinate System

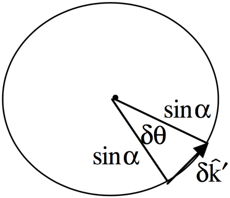
- $D_a\hat{k}'/Dt$ is the rate of change of the unit vector $\hat{k}'$ as observed in in a inertial reference frame:



- tip of $\hat{k}'(t)$ traces out a circle, α is the angle between $\vec{\Omega}$ and $\hat{k}'(t)$, and the radius of the circle $= |\hat{k}'| \sin\alpha = \sin\alpha$.
- $\delta\hat{k}' \equiv \hat{k}'(t+\delta t) - \hat{k}'(t)$ is the change in $\hat{k}'$ over a small time interval δt.

Conservation of Momentum: Rotating Coordinate System

- Let's look at the plane perpendicular to the $\vec{\Omega}$ axis



- $\vec{\Omega}$ is directed out of the center of the cirle and $|\delta\hat{k}'| = \sin\alpha\delta\theta$.

$$\left|\frac{D_a\hat{k}'}{Dt}\right| = \lim_{\delta t\to 0}\frac{\left|\delta\hat{k}'\right|}{\delta t} = \lim_{\delta t\to 0}\sin\alpha\frac{\delta\theta}{\delta t} = \Omega\sin\alpha$$

But $\left|\vec{\Omega}\times\hat{k}'\right| = \left|\vec{\Omega}\right|\left|\hat{k}'\right|\sin\alpha$, so $\left|\frac{D_a\hat{k}'}{Dt}\right| = \left|\vec{\Omega}\times\hat{k}'\right|$

Conservation of Momentum: Rotating Coordinate System

- $D_a\hat{k}'/Dt$ has direction of $\delta\hat{k}'$ as $\delta t \to 0$
- $\delta\hat{k}'$ has direction of $\vec{\Omega} \times \hat{k}'$ ($\delta\hat{k}'$ is perpendicular to $\vec{\Omega}$ and perpendicular to $\hat{k}'$ and points in the direction of increasing θ)
- Thus, $D_a\hat{k}'/Dt$ has direction $\vec{\Omega} \times \hat{k}'$
- Since $D_a\hat{k}'/Dt$ has the same magnitude and direction as $\vec{\Omega} \times \hat{k}'$, it must be $\vec{\Omega} \times \hat{k}'$
- Thus

$$\frac{D_a\hat{k}'}{Dt} = \vec{\Omega} \times \hat{k}'$$

and similarly,

$$\frac{D_a\hat{i}'}{Dt} = \vec{\Omega} \times \hat{i}' \qquad \text{and} \qquad \frac{D_a\hat{j}'}{Dt} = \vec{\Omega} \times \hat{j}'$$

Conservation of Momentum: Rotating Coordinate System

- We can now write our expression as:

$$\frac{D_a\vec{A}}{Dt} = \frac{D\vec{A}}{Dt} + A'_x\vec{\Omega}\times\hat{i}' + A'_y\vec{\Omega}\times\hat{j}' + A'_z\vec{\Omega}\times\hat{k}'$$

$$\frac{D_a\vec{A}}{Dt} = \frac{D\vec{A}}{Dt} + \vec{\Omega}\times\left(A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'\right)$$

$$\boxed{\frac{D_a\vec{A}}{Dt} = \frac{D\vec{A}}{Dt} + \vec{\Omega}\times\vec{A}}$$

- In words, this says that the total derivative as observed in a non-rotating reference frame is equal to the total derivative as observed in a rotating reference frame plus the apparent force arising from rotation.

Conservation of Momentum: Rotating Coordinate System

- Let's consider the vector form of $\vec{F} = m\vec{a}$, or $\vec{a} = \vec{F}/m$, which holds for an inertial reference frame.
- $\vec{a}$ is the acceleration observed in an inertial reference frame:

$$\vec{a} = \frac{D_a \vec{U}_a}{Dt}$$

where $\vec{U}_a = D_a \vec{r}/Dt$ is the absolute velocity of an air parcel and $\vec{r}$ is the position vector.

- Now we apply the previous relationship and set $\vec{A} = \vec{r}$

$$\frac{D_a \vec{r}}{Dt} = \frac{D\vec{r}}{Dt} + \vec{\Omega} \times \vec{r}$$

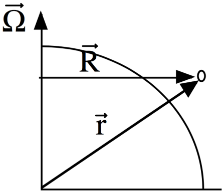
$$\vec{U}_a = \underbrace{\vec{U}}_{\text{I}} + \underbrace{\vec{\Omega} \times \vec{r}}_{\text{II}}$$

where (I) is the relative velocity of the air parcel and (II) is the velocity of solid-body rotation at location $\vec{r}$.so

Conservation of Momentum: Rotating Coordinate System

- Now let's set $\vec{A} = \vec{U}_a$:

$$
\begin{aligned}
\frac{D_a \vec{U}_a}{Dt} &= \frac{D\vec{U}_a}{Dt} + \vec{\Omega} \times \vec{U}_a \\
&= \frac{D}{Dt}\left(\vec{U} + \vec{\Omega} \times \vec{r}\right) + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \\
&= \frac{D\vec{U}}{Dt} + \vec{\Omega} \times \underbrace{\frac{D\vec{r}}{Dt}}_{\vec{U}} + \vec{\Omega} \times \vec{U} + \underbrace{\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)}_{-\Omega^2 \vec{R}}
\end{aligned}
$$



Conservation of Momentum: Rotating Coordinate System

- The final form:

$$\underbrace{\frac{D_a\vec{U}_a}{Dt}}_{1} = \underbrace{\frac{D\vec{U}}{Dt}}_{2} + \underbrace{2\vec{\Omega} \times \vec{U}}_{3} - \underbrace{\Omega^2\vec{R}}_{4}$$

1. acceleration in an inertial reference frame
2. acceleration in a non-inertial reference frame
3. Coriolis acceleration
4. centripetal acceleration

Conservation of Momentum: Rotating Coordinate System

- Plus this into Newton's 2^{nd} Law:

$$\frac{D_a\vec{U}_a}{Dt} = \frac{\vec{F}}{m}$$
$$\frac{D\vec{U}}{Dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2\vec{R} = \frac{\vec{F}}{m}$$

- Recall that $\vec{F}/m$ is the sum of all the fundamental forces per unit mass
    - PGF: $-\frac{1}{\rho}\vec{\nabla}p$
    - friction: $\nu\vec{\nabla}^2\vec{U}$
    - gravitational: $\vec{g^*}$

Conservation of Momentum: Rotating Coordinate System

- Newton's 2nd Law becomes:

$$\frac{D\vec{U}}{Dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2 \vec{R} = -\frac{1}{\rho}\vec{\nabla} p + \vec{g^*} + \nu \vec{\nabla}^2 \vec{U}$$

$$\frac{D\vec{U}}{Dt} = -\frac{1}{\rho}\vec{\nabla} p - 2\vec{\Omega} \times \vec{U} + \underbrace{\vec{g^*} + \Omega^2 \vec{R}}_{\vec{g}} + \nu \vec{\nabla}^2 \vec{U}$$

We arrive at the vector equation of motion in a rotating reference frame:

$$\boxed{\frac{D\vec{U}}{Dt} = -\frac{1}{\rho}\vec{\nabla} p - 2\vec{\Omega} \times \vec{U} + \vec{g} + \nu \vec{\nabla}^2 \vec{U}}$$

Conservation of Momentum: Rotating Coordinate System

$$\boxed{\underbrace{\frac{D\vec{U}}{Dt}}_{1} = \underbrace{-\frac{1}{\rho}\vec{\nabla}p}_{2} \underbrace{- 2\vec{\Omega} \times \vec{U}}_{3} + \underbrace{\vec{g}}_{4} + \underbrace{\nu\vec{\nabla}^2\vec{U}}_{5}}$$

1. acceleration in a rotating reference frame
2. pressure gradient force
3. Coriolis force
4. gravity
5. friction