Environmental Fluid Dynamics: Lecture 9

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Spring 2017

Overview

1. Atmospheric Dynamics: Basic Equations
    - Conservation of Momentum, continued

Atmospheric Dynamics:

Conservation of Momentum,

continued

Conservation of Momentum: Non-Inertial Reference Frame

- $\vec{F} = m\vec{a}$ is only valid for inertial (non-accelerating) reference frames.
- Note: a reference frame is not the same as a coordinate system because it depends on the motion of the observer.
- An inertial reference frame is stationary or it moves at a constant velocity.
- A non-inertial reference frame changes velocity or rotates.

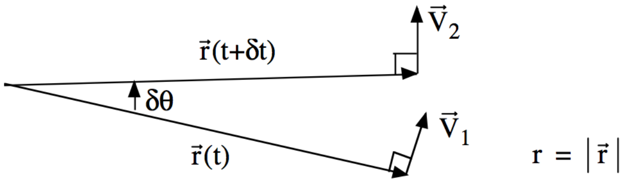
Conservation of Momentum: Non-Inertial Reference Frame

- It is convenient to work with a reference frame that is fixed with respect to Earth.
- Why? This is how we take measurements.
- The Earth rotates, so this reference frame is non-inertial.
- How do we reconcile the limitations of Newton's 2$^{\text{nd}}$ Law?
- Fortunately, we can modify $\vec{F} = m\vec{a}$ to allow for its application to non-inertial reference frames through the introduction of "apparent forces".

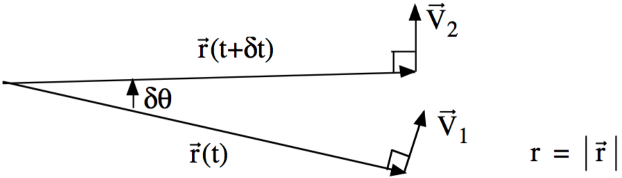
- There are two types of apparent forces that arise due to our rotating reference frame: *centrifugal* and *Coriolis*.
- **Centrifugal Force:** the inertial force on an object that is directed away from the axis of rotation that appears to act on all bodies when viewed in a rotating frame of reference.
- **Coriolis Force:** the inertial force that appears to act on an object in motion relative to a rotating frame of reference.

Conservation of Momentum: Apparent Force (Centrifugal)

- Imagine some part of the universe that is not accelerating.
- We will put a reference frame (observer) there.
- It is an inertial reference frame, so $\vec{F} = m\vec{a}$ is valid.
- Our observer sees a ball with mass m attached to a string spinning in a circle of radius r at constant angular velocity ω.
- What is ball's observed acceleration?
- Look at the ball at 2 infinitesimally close times t and $t + \delta t$.

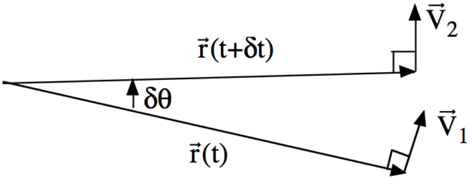


Conservation of Momentum: Apparent Force (Centrifugal)



- $\omega = d\theta/dt \rightarrow$ thus, the angular displacement $\delta\theta$ of the ball in time δt is: $\delta\theta = \omega\delta t$.
- The ball's speed $\left|\vec{V}\right| = \omega r$ is constant since ω, r are constant.
- Thus, only the direction of the ball's velocity changes.

Conservation of Momentum: Apparent Force (Centrifugal)



$r = |\vec{r}|$

- The vector change in $\vec{V}$ over a tiny time increment is $\perp$ to $\vec{V}$.

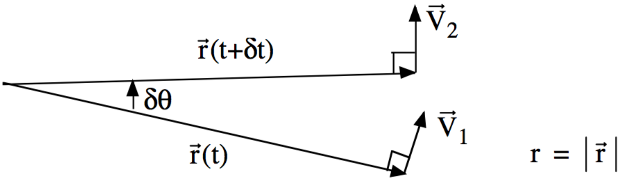
$$\left|\vec{V}\right| = \text{constant} \rightarrow \sqrt{u^2+v^2} = \text{constant} \rightarrow u^2 + v^2 = \text{constant}$$

$$\rightarrow \vec{V}\cdot\vec{V} = \text{constant} \rightarrow \frac{D}{Dt}(\vec{V}\cdot\vec{V}) = \frac{D(\text{constant})}{Dt} = 0$$

$$\rightarrow \vec{V}\cdot\frac{D\vec{V}}{Dt} + \vec{V}\cdot\frac{D\vec{V}}{Dt} = 2\vec{V}\cdot\frac{D\vec{V}}{Dt} = 0$$

Since $\vec{V} \neq 0$ and $\frac{D\vec{V}}{Dt} \neq 0$, must have $\vec{V} \perp \frac{D\vec{V}}{Dt}$

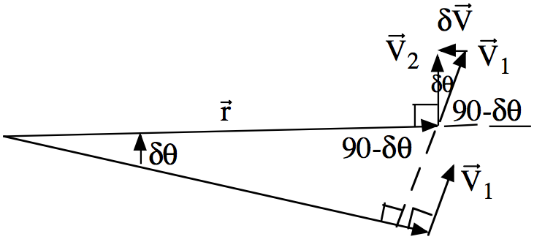
Conservation of Momentum: Apparent Force (Centrifugal)



- So, the change in $\vec{V}$ (acceleration) is perpendicular to $\vec{V}$
- Another way to think about it: there can be no change in $\vec{V}$ in the direction of $\vec{V}$ since $\left|\vec{V}\right|$ is constant. If there was such an acceleration, then $\left|\vec{V}\right|$ would increase/decrease, which is impossible since it is constant. Thus, any change in $\vec{V}$ must be in the radial direction.

Conservation of Momentum: Apparent Force (Centrifugal)

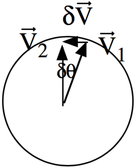
- We can see it graphically



For small $\delta\theta$, $\delta\vec{V}$ is perpendicular to $\vec{V}$ ($\vec{V_1}$ or $\vec{V_2}$) - meaning it points toward the axis of rotation ($-\hat{r}$ direction).

Conservation of Momentum: Apparent Force (Centrifugal)

- In the previous example, consider a circle with radius $\left|\vec{V}\right|$



$$\left|\delta\vec{V}\right| = \left|\vec{V}\right|\delta\theta = -\omega r\delta\theta \rightarrow \delta\vec{V} = -\omega r\delta\theta\hat{r}$$

divide bt δt

$$\frac{\delta\vec{V}}{\delta t} = -\omega r\frac{\delta\theta}{\delta t}\hat{r}$$

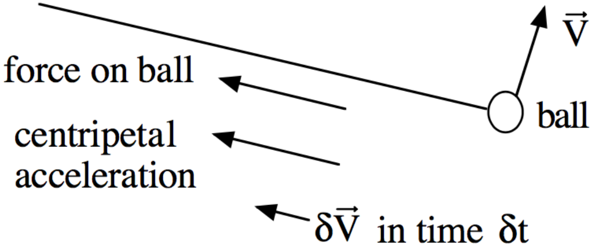
Take the limit as $\delta t \rightarrow 0$

$$\frac{D\vec{V}}{Dt} = -\omega r\frac{D\theta}{Dt}\hat{r} = -\omega^2 r\hat{r} = -\omega^2\vec{r}$$

Thus, the acceleration of the ball is inward toward to the axis of rotation and is called the centripetal acceleration.

Conservation of Momentum: Apparent Force (Centrifugal)

- The force causing this centripetal acceleration is the string pulling inward on the ball:



- Can apply $\vec{F} = m\vec{a}$ since we are in an inertial reference frame:

$$\vec{F}_{\text{on ball due to string}} = -m\omega^2\vec{r}$$

Conservation of Momentum: Apparent Force (Centrifugal)

- Consider the same physical problem, but now our observer (reference frame) is now fixed with respect to the ball.
- The ball appears stationary in this non-inertial reference frame, so the apparent acceleration is 0.
- However, there is still a force on the ball due to the string!
- Applying Newton's 2$^{\text{nd}}$ Law in this non-inertial reference frame says $\vec{F}_{\text{on ball due to string}} = 0$ which is wrong since a force does exist.

Conservation of Momentum: Apparent Force (Centrifugal)

- To make Newton's 2nd Law work in our non-inertial reference frame we need to introduce an "apparent" force that cancels with the force on the string.
- This apparent force is called the **centrifugal force**.

$$\vec{F}_{\text{on ball due to string}} + \vec{F}_{\text{centrifugal}} = 0$$

thus,

$$\vec{F}_{\text{centrifugal}} = -\vec{F}_{\text{on ball due to string}}$$
$$\vec{F}_{\text{centrifugal}} = m\omega^2 \vec{r}$$

Conservation of Momentum: Apparent Force (Centrifugal)

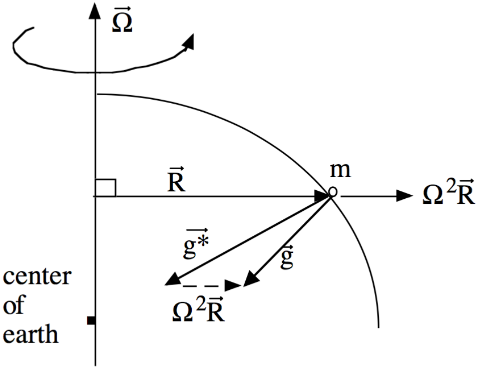
- Instead of a ball, consider a reference frame that is fixed with respect to Earth.
- Earth rotates with angular velocity $\vec{\Omega}$ ($\Omega \equiv \left|\vec{\Omega}\right|$)
- Consider a mass m at rest on the surface of Earth, $\vec{R}$ is the position vector of this mass with respect to the axis of rotation.
- We arrive at the centrifugal force per unit mass:

$$\boxed{\frac{\vec{F}_{\text{centrifugal}}}{m} = \Omega^2 \vec{R}}$$

Conservation of Momentum: Apparent Force (Centrifugal)

- We can now define the *effective gravity*, which is the sum of the fundamental gravitational force and the apparent centrifugal force:

$$\underbrace{\vec{g}}_{\text{gravity force}} = \underbrace{\vec{g^*}}_{\text{gravitational force}} + \underbrace{\Omega^2\vec{R}}_{\text{centrifugal force}}$$

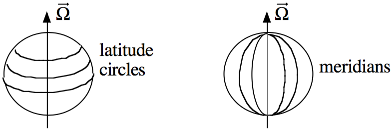


Conservation of Momentum: Apparent Force (Coriolis)

- We considered a mass at rest on Earth's surface.
- What happens if the mass is moving?
- We will need to introduce a second apparent force to enable the use of Newton's 2^{nd} Law.
- This apparent force related to movement in the rotating reference frame is named after French scientist Gaspard-Gustave de Coriolis.

Conservation of Momentum: Apparent Force (Coriolis)

- Let's define velocity in terms of Earth



- u = velocity along a latitude circle
  - $u > 0$ toward east (westerly wind)
  - $u < 0$ toward west (easterly wind)
- v = velocity along a meridian
  - $v > 0$ toward north (southerly wind)
  - $v < 0$ toward south (northerly wind)
- w = vertical velocity
  - $w > 0$ upward motion
  - $w < 0$ downward motion

Conservation of Momentum: Apparent Force (Coriolis)

- Imagine that we kick an initially resting mass m toward the east.
- Since $u > 0$ here, the mass rotates faster than earth.
- The centrifugal force on the initially resting mass was:

$$\Omega^2 \vec{R}$$

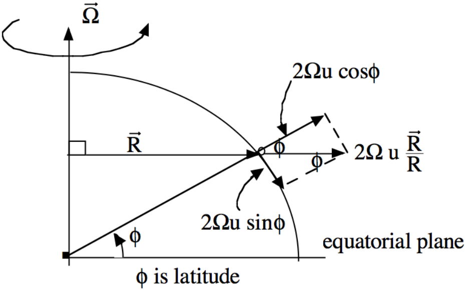
- The centrifugal force on the mass after being kicked:

$$\left(\Omega + \frac{u}{R}\right)^2 \vec{R}$$

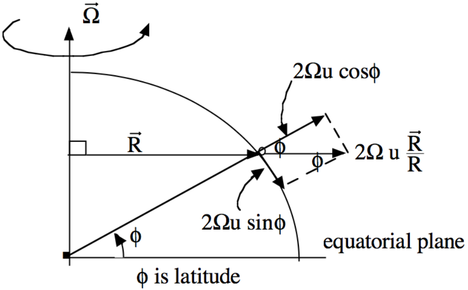
Note: velocity $=$ angular velocity $\times$ radius, so angular velocity $=$ velocity/radius

$$\left(\Omega + \frac{u}{R}\right)^2 \vec{R} = \underbrace{\Omega^2 \vec{R}}_{\text{centrifugal force}} + \underbrace{2\Omega u \frac{\vec{R}}{R}}_{\text{Coriolis force}} + \underbrace{\frac{u^2}{R^2}\vec{R}}_{\text{small, neglect}}$$

- Coriolis force in this scenario is directed radially outward from the axis of rotation.
- Coriolis has no component in latitudinal directions and projects into the meridional and vertical directions.



Conservation of Momentum: Apparent Force (Coriolis)



- Associated with this Coriolis force are the following acceleration components:

$$\left.\frac{dv}{dt}\right|_{\text{Coriolis}} = -2\Omega u \sin\phi \qquad \left.\frac{dw}{dt}\right|_{\text{Coriolis}} = 2\Omega u \cos\phi$$

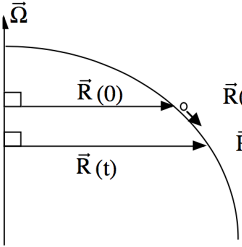
Conservation of Momentum: Apparent Force (Coriolis)

$$\left.\frac{dv}{dt}\right|_{\text{Coriolis}} = -2\Omega u \sin\phi \qquad \left.\frac{dw}{dt}\right|_{\text{Coriolis}} = 2\Omega u \cos\phi$$

- $u > 0$ (eastward): acceleration is toward the south and upward (upward Coriolis force is weak compared to gravity and slightly lessens the apparent weight of an object)
- $u < 0$ (westward): acceleration is toward the north and downward (slightly increases the apparent weight of an object)
- In either case, the Coriolis force is perpendicular to the direction of motion.
- In either case, we get a deflection to the left relative to the direction of motion.

Conservation of Momentum: Apparent Force (Coriolis)

- Imagine that we kick an initially resting mass m toward the south ($v < 0$)

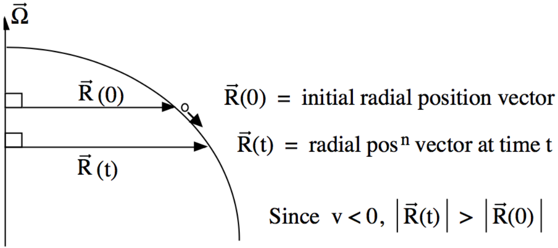


- From the conservation of angular momentum:

$$[u(t) + \Omega R(t)]\, R(t) = C$$

- Initial conditions will help us solve C.

Conservation of Momentum: Apparent Force (Coriolis)



- At the time of the kick ($t = 0$), $u(0) = 0$, and $R = R(0)$:

$$[0 + \Omega R(0)]\, R(0) = C \rightarrow C = \Omega R^2(0)$$

Thus,

$$[u(t) + \Omega R(t)]\, R(t) = \Omega R^2(0)$$

Conservation of Momentum: Apparent Force (Coriolis)

- A short time after the kick ($t = \delta t$) the mass is at radius $R(0) + \delta R$ with a southward velocity ($v < 0$). u? (we will call it δu since we expect that it will be small for small δt

$$\{\delta u + \Omega\left[R(0) + \delta R\right]\}\left[R(0) + \delta R\right] = \Omega R^2(0)$$

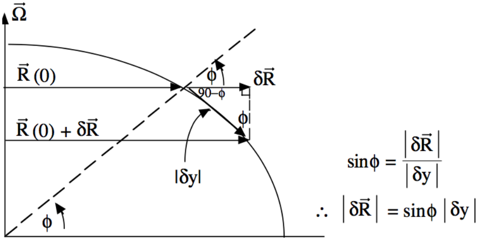
$$\delta u R(0) + \cancel{\Omega R^2(0)} + \Omega R(0)\delta R + \underbrace{\delta u \delta R}_{\text{small}} + \Omega R(0)\delta R + \underbrace{\Omega(\delta R)^2}_{\text{small}} = \cancel{\Omega R^2(0)}$$

So, we neglect $\delta u \delta R$ and $\Omega(\delta R)^2$

$$\delta u R(0) + 2\Omega R(0)\delta R = 0 \rightarrow \delta u = -2\Omega\delta R$$

- The mass develops a small westward (easterly) velocity component.
- Let's describe the acceleration.

Conservation of Momentum: Apparent Force (Coriolis)



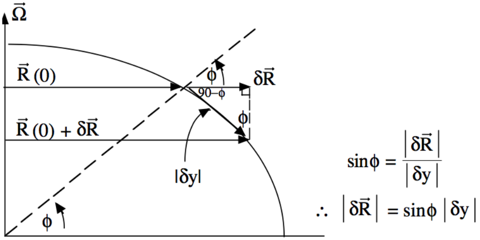
- $\delta R = -\sin\phi\delta y$ (negative since δy corresponds to a postive δR (and reverse).

$$\delta u = -2\Omega\delta R = -2\Omega(-\sin\phi\delta y) = 2\Omega\sin\phi\delta y$$

Divide by δt and take the limit as $\delta t \to 0$

$$\left.\frac{du}{dt}\right|_{\text{Coriolis}} = 2\Omega\sin\phi\frac{dy}{dt} = 2\Omega v\sin\phi$$

Conservation of Momentum: Apparent Force (Coriolis)



- For our initial southward kick ($v < 0$), $du/dt|_{\text{Coriolis}} < 0$
- u is initially 0 but becomes negative
- This means that we get a deflection toward the west (right, relative to the direction of motion)
- We get the same formula and rightward deflection if the ball is kicked north.

Conservation of Momentum: Apparent Force (Coriolis)

- Imagine that we kick an initially resting mass m upward ($w > 0$) or downward ($w < 0$).
- Conservation of angular momentum leads to:

$$\frac{du}{dt}|_{\text{Coriolis}} = -2\Omega w \cos\phi$$

- This is derived following a similar approach as for the horizontal components of momentum.

Conservation of Momentum: Apparent Force (Coriolis)

- Putting the all together:

$$\frac{du}{dt}\Big|_{\text{Coriolis}} = 2\Omega v \sin\phi - 2\Omega w \cos\phi$$
$$\frac{dv}{dt}\Big|_{\text{Coriolis}} = -2\Omega u \sin\phi$$
$$\frac{dw}{dt}\Big|_{\text{Coriolis}} = 2\Omega u \cos\phi$$

- These terms must appear as apparent forces per unit mass to allow the application of $\vec{F} = m\vec{a}$.