Environmental Fluid Dynamics

ME EN 7710 | Exam #1 | In-Class Solutions

1.) [10 points] Matching

2 isotropic
6 flux divergence
7 specific humidity
4 thermal diffusivity
1 mixing ratio

1. ratio of mass of water vapor to mass of dry air
2. invariant to rotation and reflection
3. invariant to spatial translation
4. ratio of thermal conductivity to heat capacity
5. the ability of a material to conduct heat
6. cooling of a layer due to change in net radiation with height
7. ratio of mass of water vapor to mass of moist air
8. time, ensemble, and space averages are equal

2.) [6 points] Consider a very thin, no mass, no heat capacity surface, under typical clear sky conditions. Indicate the signs of the following heat fluxes (assume a sign convention where fluxes leaving the surface are positive)

a.) Ground heat flux during the night: $-$

b.) Sensible heat flux during the day: $+$

c.) Latent heat flux during the night: $-$

3.) [4 points] Again consider the simplified surface energy balance for thin, no mass, no heat capacity surface, under typical clear sky conditions. Provide one reason why this balance might be unsatisfactory for an urban neighborhood?

Possible examples: not horizontally homogeneous, advection, non-zero heat storage, etc.

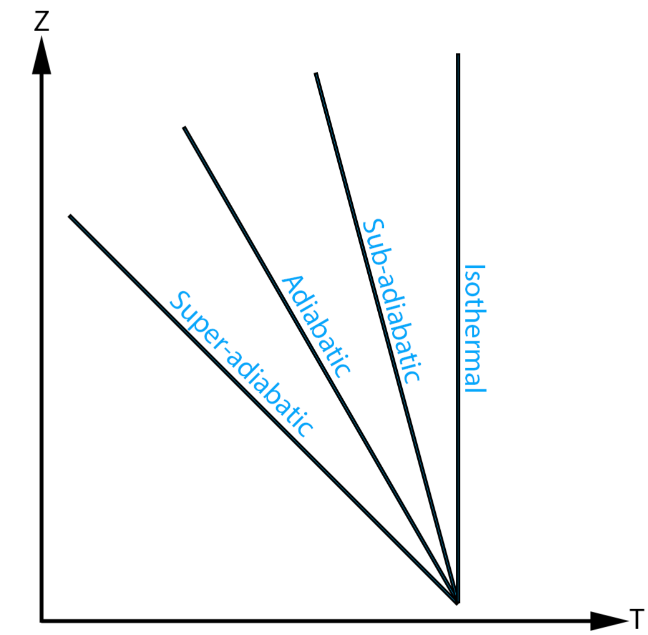
4.) [6 points] Describe the following dimensionless numbers using words or equations.

a.) Rossby Number: $\mathrm{Ro} = \frac{U}{fL}$, ratio of inertial to Coriolis forces.

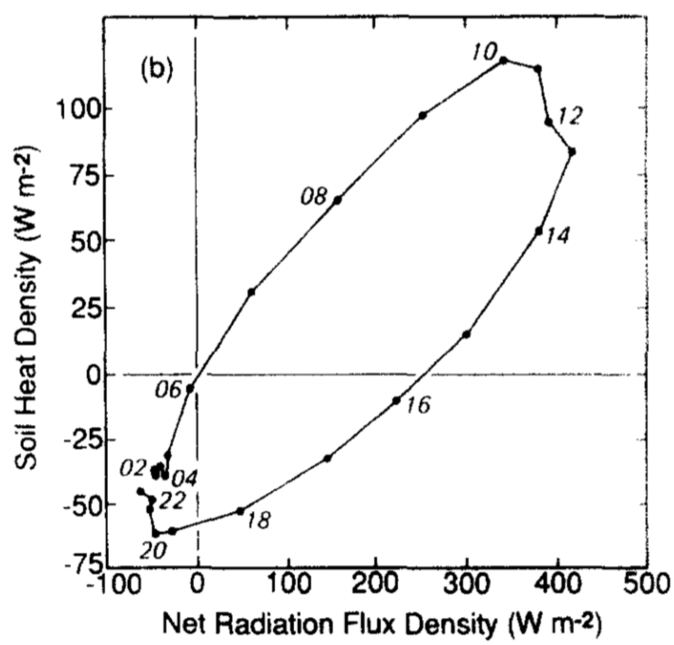
b.) Bowen Ratio: $\mathrm{BR} = \frac{H_S}{H_L}$, ratio of sensible to latent heat fluxes.

c.) Ekman Number: $\mathrm{E} = \frac{\nu}{fL^2}$, ratio of viscous to Coriolis forces.

5.) [8 points] Sketch and label the vertical temperature profiles for an (a) adiabatic, (b) superadiabatic, (c) subadiabatic, and (d) isothermal atmosphere.



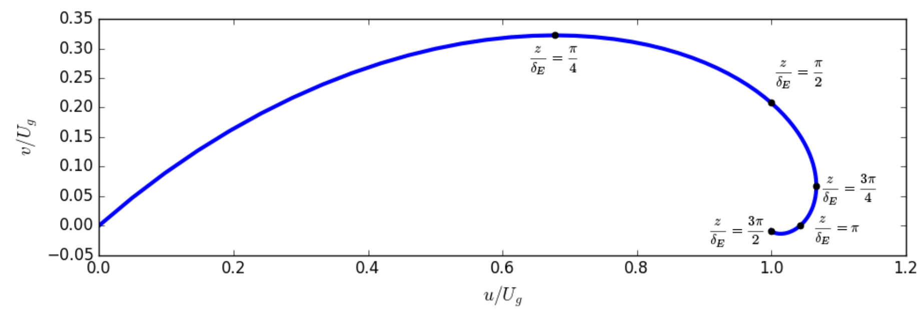
6.) [4 points] What phenomena does this figure illustrate? What is its interpretation?.



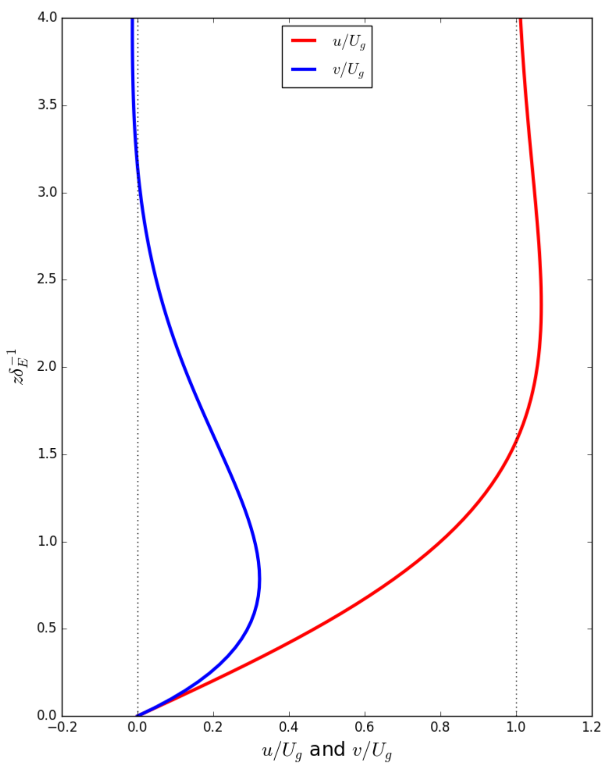
This depicts hysteresis. In this case, we see a lag between soil heat flux and net all-wave radiation.

7.) [12 points] Ekman Layer

a.) Sketch the hodograph of the Ekman layer solution above a rigid surface.



b.) Sketch the vertical profiles of the horizontal momentum components for the Ekman layer solution above a rigid surface.



c.) What is Ekman pumping?

At low levels, friction induces a flow component toward low pressure. As a result, we get horizontal convergence into the low-pressure zone. This results in rising motion (from mass conservation). This transport is called Ekman pumping.

8.) [16 points] Scale Analysis

a.) [5 points] Using characteristic values for the synoptic scale (located on the provided equation sheet), perform scale analysis of the following horizontal equation of motion.

$$\frac{\partial u}{\partial t} \quad +u\frac{\partial u}{\partial x} \quad +v\frac{\partial u}{\partial y} \quad +w\frac{\partial u}{\partial z} \quad = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad +fv \quad +\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{V}{T} \quad \frac{VV}{L} \quad \frac{VV}{L} \quad \frac{WV}{H} \quad \frac{\Delta p}{\rho L} \quad fV \quad \nu\left(\frac{V}{L^2}+\frac{V}{L^2}+\frac{V}{H^2}\right)$$

$$\frac{10}{10^5} \quad \frac{10\times 10}{10^6} \quad \frac{10\times 10}{10^6} \quad \frac{0.1\times 10}{10^4} \quad \frac{10^3}{1\times 10^6} \quad 10^{-4}\times 10 \quad 10^{-5}\left(\frac{10}{10^{12}}+\frac{10}{10^{12}}+\frac{10}{10^8}\right)$$

$$10^{-4} \quad 10^{-4} \quad 10^{-4} \quad 10^{-4} \quad \mathbf{10^{-3}} \quad \mathbf{10^{-3}} \quad (10^{-16}+10^{-16}+10^{-12})$$

b.) [3 points] Which approximation may be used to describe the horizontal flow based on the dominant terms above? Be sure to describe the relevant balance of forces.

This is the geostrophic approximation, which is the balance between the horizontal pressure gradient force and Coriolis force.

c.) [5 points] Repeat the scale analysis of the following vertical equation of motion using characteristic values for the synoptic scale.

$$\frac{\partial w}{\partial t} \quad +u\frac{\partial w}{\partial x} \quad +v\frac{\partial w}{\partial y} \quad +w\frac{\partial w}{\partial z} \quad = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad -g \quad +\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{W}{T} \quad \frac{VW}{L} \quad \frac{VW}{L} \quad \frac{WW}{H} \quad \frac{\Delta p}{\rho H} \quad g \quad \nu\left(\frac{W}{L^2}+\frac{W}{L^2}+\frac{W}{H^2}\right)$$

$$\frac{0.1}{10^5} \quad \frac{10\times 0.1}{10^6} \quad \frac{10\times 0.1}{10^6} \quad \frac{0.1\times 0.1}{10^4} \quad \frac{10^5}{1\times 10^4} \quad 10 \quad 10^{-5}\left(\frac{0.1}{10^{12}}+\frac{0.1}{10^{12}}+\frac{0.1}{10^8}\right)$$

$$10^{-6} \quad 10^{-6} \quad 10^{-6} \quad 10^{-6} \quad \mathbf{10} \quad \mathbf{10} \quad (10^{-18}+10^{-18}+10^{-14})$$

d.) [3 points] Which approximation may be used to describe the vertical flow based on the dominant terms above? Be sure to describe the relevant balance of forces.

This is the hydrostatic approximation, which is the balance between the vertical pressure gradient force and gravity.

9.) [5 points] Using the provided Coriolis term that we derived for the mechanical energy equation, show the work done by the Coriolis force.

$$\vec{U} \cdot \left(-2\rho\vec{\Omega} \times \vec{U}\right) = \epsilon_{ij3} f u_i u_j$$

Using the rules of the alternating unit tensor, you can show this is equal to zero.

$$\begin{aligned}
i = 1, j = 1 \rightarrow \quad & \epsilon_{113} fuu = & 0 \\
i = 1, j = 2 \rightarrow \quad & \epsilon_{123} fuv = & fuv \\
i = 1, j = 3 \rightarrow \quad & \epsilon_{133} fuw = & 0 \\
i = 2, j = 1 \rightarrow \quad & \epsilon_{213} fvu = & -fuv \\
i = 2, j = 2 \rightarrow \quad & \epsilon_{223} fvv = & 0 \\
i = 2, j = 3 \rightarrow \quad & \epsilon_{233} fvw = & 0 \\
i = 3, j = 1 \rightarrow \quad & \epsilon_{313} fwu = & 0 \\
i = 3, j = 2 \rightarrow \quad & \epsilon_{323} fwv = & 0 \\
i = 3, j = 3 \rightarrow \quad & \epsilon_{333} fww = & 0
\end{aligned}$$

Thus, $\epsilon_{ij3} f u_i u_j = fuv - fuv = 0$.

Alternatively, we showed that $-2\rho\Omega \times \vec{U}$ is $\perp$ to $\vec{U}$, and thus $\vec{U} \cdot \left(-2\rho\vec{\Omega} \times \vec{U}\right) = 0$.

10.) [3 points] What are the three typical approaches to studying turbulence?

a.) Analytically

b.) Direct/numerically

c.) Statistically

11.) [3 points] Name three characteristics of turbulence.

Could include: unsteady, intermittent, scale-free behavior, random, 3D, high Re, large vorticity, mixing effect, decays without input, non-linear.

12.) [3 points] What is meant by the ergodic condition?

Time, space, and ensemble averages are equivalent.

13.) [14 points] Taylor-Proudman Theorem

a.) [10 points] Starting with the simplified equations of motion for a rotating, inviscid, homogeneous fluid, derive the Taylor-Proudman outcome of $\partial \vec{U}/\partial z = 0$

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (1) \qquad 2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (2) \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad (3)$$

First take $\partial/\partial y$ of Eq. (1):

$$-2\Omega\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial y}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial y}$$

Next take $\partial/\partial x$ of Eq. (2):

$$2\Omega\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial y}$$

Both equations are equal:

$$-2\Omega\frac{\partial v}{\partial y} = 2\Omega\frac{\partial u}{\partial x} \rightarrow 2\Omega\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Recall that the incompressibility condition says $\vec{\nabla}\cdot\vec{U} = 0$. Therefore, $\partial w/\partial z = 0$.

Next, differentiate Eqs. (1) and (2) with respect to z:

$$-2\Omega\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\frac{\partial p}{\partial z}$$
$$2\Omega\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial p}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial y}\frac{\partial p}{\partial z}$$

Using Eq. (3):

$$-2\Omega\frac{\partial v}{\partial z} = \frac{\partial g}{\partial x} = 0 \qquad 2\Omega\frac{\partial u}{\partial z} = \frac{\partial g}{\partial y} = 0$$

Both equations are equal:

$$2\Omega\frac{\partial v}{\partial z} = 2\Omega\frac{\partial u}{\partial z} \rightarrow \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

We already showed that $\partial w/\partial z = 0$, so

$$\frac{\partial \vec{U}}{\partial z} = 0$$

b.) [4 points] Explain one implication of the theorem.

- This outcome shows that the velocity vector does not vary in the direction of the $\vec{\Omega}$.
- In other words, steady, slow motions in a rotating, inviscid, homogeneous fluid are two-dimensional.
- For the case of a rotating steady, inviscid, homogeneous fluid, Taylor's experiments showed that bodies moving parallel or perpendicular to the axis of rotation carry with them a Taylor column.
- This Taylor column of fluid is oriented parallel to the axis of rotation.
- This phenomenon is similar to horizontal solid-body blocking in the real (stratified) world, such as flow encountering a mountain.

14.) [6 points] Consider the following equation for the internal energy (I, we also wrote this as e) at a point in the atmosphere. We derived this as part of the thermal energy equation. Provide a physical meaning for each term.

$$\underbrace{\rho\frac{DI}{Dt}}_{1} = \underbrace{-\vec{\nabla}\cdot\vec{q}}_{2} \underbrace{-\vec{\nabla}\cdot(p\vec{U})}_{3} \underbrace{-\vec{\nabla}\cdot\overrightarrow{R_n}}_{4} + \underbrace{L_v\epsilon}_{5} + \underbrace{\mu\Phi_\nu}_{6}$$

1. Rate of gain of internal energy
2. Rate of internal energy input by conduction
3. Reversible rate of internal energy increase by compression
4. Rate of internal energy input by net radiation
5. Rate of internal energy input by phase change
6. Irreversible rate of internal energy increase by viscous dissipation

Potentially Useful Information

Characteristic Values for the Synoptic Scale

- $\nu \sim 10^{-5}\ \mathrm{m}^2\,\mathrm{s}^{-1}$
- $V \sim 10\ \mathrm{ms}^{-1}$
- $W \sim 0.1\ \mathrm{ms}^{-1}$
- $L \sim 1000\ \mathrm{km} = 10^6\ \mathrm{m}$
- $H \sim 10\ \mathrm{km} = 10^4\ \mathrm{m}$
- $T \sim L/V \sim 10^5\ \mathrm{s}$
- $f \sim 10^{-4}\ \mathrm{s}^{-1}$
- $\rho \sim 1\ \mathrm{kg\,m}^{-3}$
- Δp in horizontal $\sim 10\ \mathrm{mb} = 1000\ \mathrm{Pa}$
- Δp over vertical length scale $H \sim 1000\ \mathrm{mb} = 10^5\ \mathrm{Pa}$

Tensors

$$\delta_{mn} = \begin{cases} +1, & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\epsilon_{mnq} = \begin{cases} +1, & \text{if } mnq = 123, 231, 312 \quad \text{even permutation} \\ -1 & \text{if } mnq = 321, 213, 132 \quad \text{odd permutation} \\ 0 & \text{if } m = n, n = q, q = m \quad \text{any two indices repeated} \end{cases}$$