ENERGY CASCADE IN LARGE-EDDY SIMULATIONS OF TURBULENT FLUID FLOWS

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1. INTRODUCTION

Computer simulations of three-dimensional turbulent flows which explicitly account for the motions of eddies ranging in size down to the inertial subrange are now possible. In most cases of interest, motions on the order of the dissipation length scale cannot be treated explicitly. Modifications of the Navier–Stokes equations must then be introduced to simulate properly the energy cascade. Considerable "damming up" of the turbulence energy in the large scales would occur, for example, if the unmodified equations were used with an energy-conserving finite-difference scheme on the advective term.

One approach to the problem is to use an eddy viscosity to account for the influence of the subgrid-scale motions on the large-scale fluctuations (Lilly, 1967). In this model, the energy cascade is then viewed solely as an energy loss of the large-scales due to an artificial viscosity arising from subgrid-scale motions. The advective term for the large-scale motions is unmodified.

In this paper, the derivation of smoothed or filtered momentum and continuity equations for the large-scale, energy-containing eddies is reexamined. Noting that the large-scale motions vary in a nonnegligible way over an averaging volume, we investigate a more accurate, modified advective term in the momentum equations for these motions. This term is nonconservative and is shown to lead to significant energy extraction from the large scales due to triple correlations of these motions. The subgrid-scale Reynolds stress term is still present but plays a reduced role as far as the energy cascade is concerned. Similar arguments are applied to the analysis of large-scale fluctuations of a passive scalar.

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2. Problem of Numerical Simulation

We consider an incompressible flow whose time evolution is given by the Navier–Stokes and continuity equations for the velocity components $u_i(\mathbf{x}, t)$, $i = 1, 2, 3$ and the pressure $p(\mathbf{x}, t)$:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i, \tag{2.1}$$

$$\partial u_i/\partial x_i = 0. \tag{2.2}$$

These equations, along with appropriate initial and boundary conditions, will yield the flow field for all later times (although for turbulent flows this field is likely to be unstable with respect to small perturbations in the initial or boundary conditions). Due to the wide range of length scales present in real turbulent flows, however, the full numerical simulation of such flows is not yet possible, in general. The required number of mesh points on a three-dimensional grid is proportional to $\mathrm{Re}^{9/4}$ where Re is the Reynolds number (Hirt, 1969). For Re = 50,000 about 10^9 mesh points would be required to simulate all the turbulent eddies down to and including those with a dissipation length scale. On the other hand, the present capability of one of the largest available machines (ILLIAC IV) is about 10^6 mesh points. The most ambitious simulation reported to date in terms of total number of mesh points is a study by Orszag and Patterson (1972) of three-dimensional homogeneous isotropic turbulence using approximately $(32)^3$ mesh points in Fourier space. The Reynolds number based on Taylor microscale was $R_\lambda = 35$, within the range of wind tunnel experiments.

In most situations, however, full simulations are not practical, or even possible. On the other hand, most of the momentum transport and turbulent diffusion is carried out by the large-scale energy-containing eddies. Therefore, simulation of these large-scale fluctuations is often of great interest. Hence, we turn to the problem of deriving momentum and continuity equations for these large-scale turbulent fluctuations.

3. Filtered Momentum and Continuity Equations

If $f(\mathbf{x})$ is a function containing all the scales we define, quite generally, the large-scale or resolvable-scale component of f to be denoted $\bar{f}$ and given by a convolution of f with a filter function $G(\mathbf{x})$,

$$\bar{f}(\mathbf{x}) = \int G(\mathbf{x} - \mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}'. \tag{3.1}$$

Integration is over the flow volume. Some examples of filters are shown in Fig.1. The one shown in Fig. 1c corresponds to a truncated Fourier expansion with $|k_i| < \pi/\Delta$. The other two are more localized in the spatial variables and are representative of finite-difference schemes based, for example, on expansions in terms of piecewise continuous polynomials. The filter shown in Fig. 1a was used by Lilly (1967).

Note that by integration by parts we find that

$$\overline{\partial f/\partial x_i} = \partial(\bar{f})/\partial x_i, \tag{3.2}$$

if f vanishes on the boundaries. Filtering Eqs. (2.1) and (2.2) therefore gives

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_i u_j}) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \nabla^2 \bar{u}_i \tag{3.3}$$

$$\partial \bar{u}_i/\partial x_i = 0. \tag{3.4}$$

To avoid writing dynamical equations for $\overline{u_i u_j}$, we must approximate it in terms of combinations of the $\bar{u}_k$ and their derivatives.

If we decompose u_i into its resolvable-scale and subgrid-scale components, $u_i = \bar{u}_i + u_i'$, then

$$\overline{u_i u_j} = \overline{\bar{u}_i \bar{u}_j} - \tau_{ij} + \tfrac{1}{3}\eta_{kk}\delta_{ij} \tag{3.5}$$

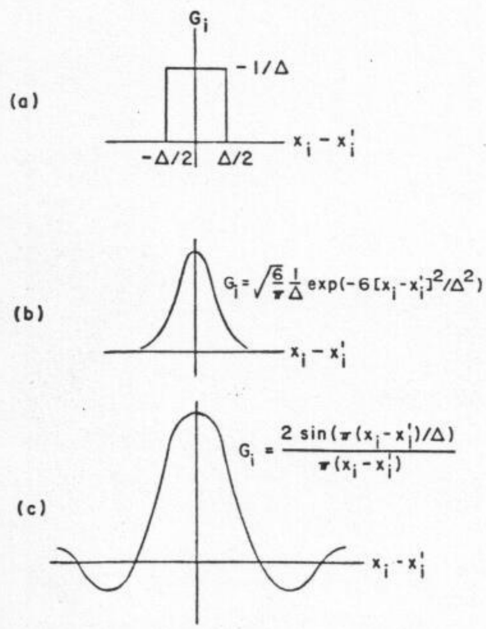


FIG. 1. Possible spatial filters defining large-scale quantities with $G = G_1 G_2 G_3$. The filters of (a) and (b) have identical second moments. The filter of (c) is equivalent to the finite Fourier expansion method.

where

$$\tau_{ij} = -(\eta_{ij} - \tfrac{1}{3}\eta_{kk}\delta_{ij}) \tag{3.6}$$

$$\eta_{ij} = \overline{u_i'\overline{u}_j} + \overline{\overline{u}_i u_j'} + \overline{u_i' u_j'}. \tag{3.7}$$

The averaged momentum and continuity equations become

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{\overline{u}_i\overline{u}_j}) = -\frac{\partial}{\partial x_i}\left(\frac{\overline{p}}{\rho} + \frac{1}{3}\eta_{kk}\right) + \frac{\partial \tau_{ij}}{\partial x_j} + \nu\,\nabla^2\overline{u}_i \tag{3.8}$$

$$\partial\overline{u}_i/\partial x_i = 0. \tag{3.9}$$

To proceed one must model τ_{ij} in terms of the $\overline{u}_k$. The function η_{kk} appearing in Eq. (3.8) may be combined with $\overline{p}$ and therefore need not be calculated explicitly.

The usual approach (Lilly, 1967) is to approximate

$$\overline{\overline{u}_i\overline{u}_j} \simeq \overline{u}_i\overline{u}_j \tag{3.10}$$

(or to lump the difference into the definition of η_{ij}) and model τ_{ij} by an eddy viscosity hypothesis,

$$\tau_{ij} = K\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right), \tag{3.11}$$

where K is an eddy viscosity coefficient, variable in space and time. Lilly (1967) has shown that if K is taken to be similar to an expression used by Smagorinsky (1963),

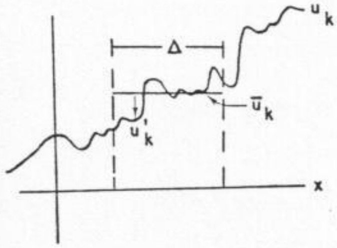
$$K = (c\Delta)^2\left[\frac{\partial \overline{u}_j}{\partial x_i}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)\right]^{1/2} \tag{3.12}$$

where Δ is the mesh spacing (or width of the G function), then the resultant energy dissipation of the large scales is consistent with the Kolmogorov power spectrum. Furthermore, the constant c is dependent only on Kolmogorov's universal constant α. Deardorff (1970) has used this approach to simulate turbulent channel flow with some success but found that the eddy viscosity constant c had to be chosen somewhat lower than that calculated by Lilly, otherwise the turbulence was excessively damped out (see also Deardorff, 1971).

In a recent simulation of an atmospheric boundary layer Deardorff (1973) abandoned the above eddy viscosity model and resorted to developing dynamical equations for the subgrid Reynolds' stresses and other relevant subgrid fluxes. The presence of a stably stratified layer apparently could not be accommodated with the use of an eddy coefficient. Perhaps the use of the modified or filtered advective term $\partial(\overline{\overline{u}_i\overline{u}_j})/\partial x_j$, i.e., avoiding the use of the approximation (3.10), would have remedied the situation. It is shown below that the filtered term plays an important role in the energy extraction from the large scales whereas the unfiltered term, $\partial(\bar{u}_i\bar{u}_j)/\partial x_j$ is energy-conserving up to finite-differencing errors.

The implications of the assumption $\overline{\bar{u}_i\bar{u}_j} = \bar{u}_i\bar{u}_j$ are illustrated in Fig. 2. This assumption is satisfied if the $\bar{u}_k$ remain constant over an averaging volume (Fig. 2a). One might compensate by dealing with a subgrid component u_k' which is effectively larger than that obtained when the variation of $\bar{u}_k$ over an averaging volume is explicitly accounted for (Fig. 2b). In the former case the modeling of the subgrid terms is clearly more critical. An exceptional case is the truncated Fourier expansion (filter of Fig. 1c) where the difference between $\overline{\bar{u}_i\bar{u}_j}$ and $\bar{u}_i\bar{u}_j$ is identically zero in the dynamical equations for the large-scale flow. We comment further on this case in the next section.

(a)



(b)

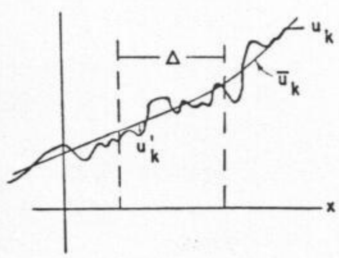


FIG. 2. Two possible definitions of the subgrid-scale component u_k'.

4. Energy Loss of the Large-Scale Turbulence

In the above model, all the energy dissipation of the large scales is viewed as a result of Reynolds stress of the subgrid-scale turbulence and modeled by an eddy viscosity times the squared deformation tensor of the large-scale flow. However, a different mechanism appears to be responsible for a substantial portion of the large-scale dissipation arising from the fact that $\overline{\bar{u}_i\bar{u}_j} - \bar{u}_i\bar{u}_j$ is not generally negligible as discussed above.

Assuming that the influence of the molecular viscosity term is negligible, energy loss of the large scales will occur through the action of the nonlinear, resolvable-scale and subgrid scale terms $\partial(\overline{\bar{u}_i\bar{u}_j})/\partial x_j$ and $\partial\tau_{ij}/\partial x_j$, respectively. The rate of energy cascade due to the former we denote ε_{RS} and to the latter, ε_{SGS}. If ε is the total loss rate, then

$$\varepsilon = \varepsilon_{RS} + \varepsilon_{SGS} \tag{4.1}$$

Multiplying Eq. (3.8) by $\bar{u}_i$ and volume averaging we find that

$$\varepsilon_{RS} = \langle \bar{u}_i \, \partial(\overline{\bar{u}_i\bar{u}_j})/\partial x_j \rangle \tag{4.2}$$

$$\varepsilon_{SGS} = \langle \bar{u}_i \, \partial\tau_{ij}/\partial x_j \rangle \tag{4.3}$$

where $\langle \; \rangle$ denotes volume averaging

$$\langle h \rangle = V^{-1} \int_V h(\mathbf{x}) \, d\mathbf{x} \tag{4.4}$$

We concentrate on the evaluation of ε_{RS}. In terms of the triple correlation tensor, defined by

$$\bar{S}_{ik,j}(\xi) = \langle \bar{u}_i(\mathbf{x})\bar{u}_k(\mathbf{x})\bar{u}_j(\mathbf{x} + \xi) \rangle, \tag{4.5}$$

(an overbar denotes a quantity corresponding to the filtered flow field $\bar{u}_i$) ε_{RS} can be written as

$$\varepsilon_{RS} = -\int G(\xi) \frac{\partial}{\partial \xi_j} \bar{S}_{ij,i}(\xi) \, d\xi \tag{4.6}$$

Because of the presence of the filter G, the behavior of $\bar{S}_{ij,i}$ only in the domain $|\xi| \lesssim \Delta$ is important. We assume that the velocity fluctuations in this range of scales are homogeneous and isotropic in which case $\bar{S}_{ij,k}(\xi)$ can be written in terms of a single scalar function $\bar{k}(r)$ $(r = |\xi|)$ (Hinze, 1959).

$$\bar{S}_{ik,j}(\xi) = \bar{q}^3 \left[\left(\bar{k} - r\frac{d\bar{k}}{dr} \right) \frac{\xi_i \xi_k \xi_k}{2r^3} - \frac{\bar{k}}{2} \delta_{ij} \frac{\xi_j}{r} + \frac{1}{4r} \frac{d(r^2\bar{k})}{dr} \left(\delta_{ij} \frac{\xi_k}{r} + \delta_{kj} \frac{\xi_i}{r} \right) \right], \tag{4.7}$$

where $\bar{q}^2$ is the filtered turbulence intensity

$$\bar{q}^2 = \langle \bar{u}_i \bar{u}_i \rangle. \tag{4.8}$$

Performing the differentiations and contractions required by (4.6), we find that

$$\varepsilon_{RS} = -\bar{q}^3 \int G(\xi) \bar{f}(|\xi|) \, d\xi, \tag{4.9}$$

where

$$\bar{f}(r) = \frac{1}{2r^2}\frac{d}{dr}\left(r^3\frac{d\bar{k}}{dr} + 4r^2\bar{k}\right). \tag{4.10}$$

For a spherically symmetric filter, (4.9) reduces to

$$\varepsilon_{RS} = -4\pi\bar{q}^3\int_0^\infty G(r)\bar{f}(r)r^2\,dr. \tag{4.11}$$

To proceed further we must obtain at least an approximate form for $\bar{k}(r)$. This function is a scalar triple correlation having the basic definition

$$\bar{k}(r) = \langle\bar{u}_1^2(\mathbf{x})\bar{u}_1(\mathbf{x} + \hat{e}_1 r)\rangle/\bar{q}^3. \tag{4.12}$$

From symmetry requirements and the fact that

$$\langle\bar{u}_1^2\,\partial\bar{u}_1/\partial x\rangle = \tfrac{1}{3}\langle\partial\bar{u}_1^3/\partial x_1\rangle = 0,$$

one can show that $\bar{k}$ has the small r expansion (Hinze, 1959),

$$\bar{k}(r) = \frac{\bar{k}_0'''r^3}{3!} + \frac{\bar{k}_0^{(v)}r^5}{5!} + \cdots; \tag{4.13}$$

this yields for $\bar{f}(r)$ the expansion

$$\bar{f}(r) = \tfrac{35}{12}\bar{k}_0'''r^2 + \tfrac{21}{80}\bar{k}_0^{(v)}r^4 + \cdots. \tag{4.14}$$

Assuming that the $\bar{u}_i$ fluctuations include a portion of the inertial subrange and the smallest scale in the subrange is $\approx \Delta$, then $\bar{k}(r)$ is linear in that subrange with a known coefficient (Kolmogorov, 1941),

$$\bar{k}(r) = -2\varepsilon r/15\bar{q}^3 \qquad (\Delta \ll r \ll L_0) \tag{4.15}$$

where L_0 is the scale of the energy-containing eddies. The resultant behavior of $\bar{f}(r)$ is

$$\bar{f}(r) = -\varepsilon/\bar{q}^3 \qquad (\Delta \ll r \ll L_0). \tag{4.16}$$

Note that using the inertial subrange form of $\bar{f}(r)$ in (4.9) yields $\varepsilon_{RS} = \varepsilon$. This is no accident. In fact $2\bar{q}^3 f(r)$ represents the advective term in the Kármán–Howarth equation (von Kármán and Howarth, 1938), a dynamical equation for the correlation $\bar{Q}_{ii}(r) = \langle\bar{u}_i(x)\bar{u}_i(x + \hat{e}_1 r)\rangle$, and in the inertial subrange of the filtered flow this term is solely responsible for the energy loss of $\bar{Q}_{ii}(r)$. Thus $2q\bar{f}(r) = -2\varepsilon$ for $\Delta \ll r \ll L_0$. Equation (4.15) then follows by using (4.10).

The corrections to (4.16) for small r however will give $\varepsilon_{RS} < \varepsilon$. This is shown qualitatively in Fig. 3. The behavior indicated by curve A assumes that the quadratic term $\frac{35}{12}\bar{k}_0'''r^2$ dominates the small r behavior up to the

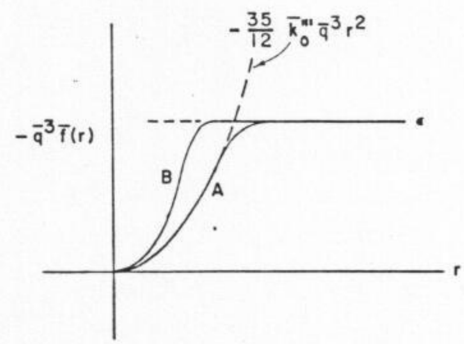


FIG. 3. Two possible interpolations of $\bar{k}(r)$ between known asymptotic forms.

inertial subrange asymptote. Unpublished calculations based on turbulence as given by Burgers' equation (Burgers, 1940, or see Burgers, 1948) ("Burgerlence") suggest that $\bar{k}(r)$ will approach its asymptotic form more quickly as shown by curve B.

Furthermore, turbulence measurements in an atmospheric boundary layer (McConnell, 1973) and an air jet (Clay, 1973) have shown that the inertial subrange behavior of $k(r)$ given by (4.15) persists down to about a Kolmogorov length η whereas curve A assumes a lower bound on the linear range of $k(r)$ at $\simeq 5\eta$.

Nevertheless, as a tentative lower bound to the amount of energy loss attributable to ε_{RS} we will use curve A for $\bar{k}(r)$. With the filter

(4.17) $$G(\xi) = [\Delta^{-1}(6/\pi)^{1/2}]^3 \exp(-6|\xi|^2/\Delta^2),$$

(4.9) gives

(4.18) $$\varepsilon_{RS} = -\tfrac{35}{48}\bar{q}^3\bar{k}_0'''\Delta^2$$

plus a negligible contribution from the asymptotic form of $\bar{f}(r)$.

We relate $\bar{k}_0'''$ to skewness by noting that

(4.19) $$\begin{aligned}\langle(\partial\bar{u}_1/\partial x_1)^3\rangle &= \langle \bar{u}_1\bar{u}_1\ \partial^3\bar{u}_1/\partial x_1^3\rangle \\ &= \frac{\partial^3}{\partial\xi_1^3}S_{11,1}(\xi)\Big|_{=0} \\ &= \bar{q}^3\bar{k}_0''',\end{aligned}$$

and therefore

(4.20) $$\begin{aligned}\varepsilon_{RS} &= -\tfrac{35}{48}\langle(\partial\bar{u}_1/\partial x_1)^3\rangle\Delta^2 \\ &= \tfrac{35}{48}\bar{S}\langle(\partial\bar{u}_1/\partial x_1)^2\rangle^{3/2}\Delta^2,\end{aligned}$$

where $\bar{S}$ is the skewness defined by

$$\bar{S} = \left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^3 \right\rangle \Big/ \left\langle \left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 \right\rangle^{3/2}. \tag{4.21}$$

Following Lilly (1967) we complete the calculation by relating $\langle(\partial u_1/\partial x_1)^2\rangle$ to the Kolmogorov spectrum. In the initial range the energy spectrum of turbulence is

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}. \tag{4.22}$$

However, the filtering process will truncate the high wave number end of this spectrum so that the spectrum of the filtered turbulence is

$$\bar{E}(\mathbf{k}) = \alpha\varepsilon^{2/3}k^{-5/3}\,|\,\hat{G}(\mathbf{k})\,|^2, \tag{4.23}$$

where $\hat{G}(\mathbf{k})$ is the Fourier transform of the filter function $G(\mathbf{x})$,

$$\hat{G}(\mathbf{k}) = \int e^{i\mathbf{k}\cdot\mathbf{x}}G(\mathbf{x})\,d\mathbf{x}. \tag{4.24}$$

Of particular interest is the integral

$$\int_0^\infty k^2\bar{E}(k)\,dk \int |\,\hat{G}(\mathbf{k})\,|^2\,d\Omega_k/4\pi = -\tfrac{1}{2}\langle\bar{u}_i\nabla^2\bar{u}_i\rangle = \tfrac{15}{2}\langle(\partial\bar{u}_1/\partial x_1)^2\rangle, \tag{4.25}$$

where again we have used isotropy in the final step. Combining (4.20), (4.23), and (4.25), ε_{RS} takes the form

$$\varepsilon_{RS} = \frac{35}{48}\bar{S}\left[\frac{2\alpha}{15}\int_0^\infty k^{1/3}\,dk \int |\,\hat{G}(\mathbf{k})\,|^2\frac{d\Omega_k}{4\pi}\right]^{3/2}\Delta^2\varepsilon. \tag{4.26}$$

For the Gaussian filter (4.17) we have

$$\hat{G}(\mathbf{k}) = \exp(-\Delta^2k^2/24) \tag{4.27}$$

Using $\alpha = 1.62$ (Wyngaard and Pao, 1972), we obtain

$$\varepsilon_{RS} = 0.49\bar{S}\varepsilon. \tag{4.28}$$

Experimentally determined values of skewness vary slightly, depending on Reynolds number, from $S \approx 0.40$ for wind-tunnel grid turbulence ($R_\lambda = 50$–100) to $S = 0.60$–0.85 for atmospheric turbulence ($R_\lambda \approx 10^3$–10^4) (Wyngaard and Pao, 1972). Thus, without more information on the behavior of $\bar{k}(r)$ we obtain the tentative lower bound for ε_{RS}

$$\varepsilon_{RS} \gtrsim 0.3\varepsilon \pm 0.1\varepsilon. \tag{4.29}$$

The SGS term $\partial\tau_{ij}/\partial x_j$ must account for the remainder of the losses.

As mentioned earlier, with the use of the truncated Fourier representation $\overline{\bar{u}_i\bar{u}_j}$ has the same large-scale Fourier components as does $\bar{u}_i\bar{u}_j$. Therefore, for this case,

$$\varepsilon_{RS} = \langle \bar{u}_i\, \partial(\overline{\bar{u}_i\bar{u}_j})/\partial x_j \rangle \\ = \langle \bar{u}_i\, \partial(\bar{u}_i\bar{u}_j)/\partial x_j \rangle = 0. \tag{4.30}$$

The interpretation is related to the fact that large-wave-number Fourier modes need the assistance of small wave-number modes to transfer energy from large scales to small scales. In the Fourier method the sharp cutoff in wave-number space precludes such a transfer whereas the localized spatial filters of Figs. 2a and 2b produce smooth, gradual filtering in wave-number space which evidently allows for energy transfer to the subgrid scales.

5. Turbulent Diffusion of a Passive Scalar

If the numerical simulation is to include the large-scale fluctuations of a passive scalar field $\psi(\mathbf{r}, t)$, one must consider the filtered equation of motion

$$\partial\bar{\psi}/\partial t + \overline{u_j\, \partial\psi/\partial x_j} = \kappa\nabla^2\bar{\psi} \tag{5.1}$$

where κ is the diffusivity. Decomposing u_j and ψ into large-scale and subgrid-scale components gives for the convection term,

$$\overline{u_j\, \partial\psi/\partial x_j} = \overline{\bar{u}_j\, \partial\bar{\psi}/\partial x_j} + \text{subgrid contributions} \tag{5.2}$$

Analogous to the preceding results, the filtered large-scale convection term on the right-hand side of (5.2) will produce a loss in scalar variance χ due to the mixed triple correlation $\langle \bar{\psi}(\mathrm{x})\bar{u}_1(\mathrm{x})\bar{\psi}(\mathrm{x} + \hat{e}_1 r)\rangle$. This correlation is cubic for small r (Corrsin, 1951) and linear ($\simeq \chi r/3$) in the resolvable-scale portion of the convective subrange. Again, the amount of scalar variance dissipation due to the filtered large-scale convection term depends on the extent to which this linearity penetrates into the small r regime.

6. Conclusions

Numerical simulation of all the scales of a turbulent flow, even at modest Reynolds numbers, is generally not practical. However, most information of interest can be obtained by simulating the motion of the large-scale, energy-containing eddies. The large-scale fluctuations satisfy filtered or averaged momentum and continuity equations. Averaging the nonlinear advection term yields two terms. One is the Reynolds stress contribution from the subgrid-scale turbulence and the other is the filtered advection term for the large scales,

$$\partial(\overline{\bar{u}_i\bar{u}_j})/\partial x_j\,.$$

A significant portion of the large-scale dissipation is provided by appropriate treatment of this term. In evaluating this term, the variations of $\partial(\bar{u}_i\bar{u}_j)/\partial x_j$ within an averaging volume defined by $G(\mathbf{x})$ should be explicitly accounted for. One obvious possibility is to represent $\partial(\overline{\bar{u}_i\bar{u}_j})/\partial x_j$ as a weighted average of the values of $\partial(\bar{u}_i\bar{u}_j)/\partial x_j$ at neighboring grid points. Another possibility is to use the Taylor expansion

$$\bar{u}_k(\mathbf{x}') = \bar{u}_k(\mathbf{x}) + (\mathbf{x}' - \mathbf{x}) \cdot \nabla\bar{u}_k(\mathbf{x}) + O(|\mathbf{x}' - \mathbf{x}|^2) \tag{6.1}$$

in the definition

$$\overline{\bar{u}_i\bar{u}_j} = \int G(\mathbf{x} - \mathbf{x}')\bar{u}_i(\mathbf{x}')\bar{u}_j(\mathbf{x}')\, d\mathbf{x}', \tag{6.2}$$

with the result

$$\frac{\partial(\overline{\bar{u}_i\bar{u}_j})}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j + \gamma\frac{\partial\bar{u}_i}{\partial x_l}\frac{\partial\bar{u}_j}{\partial x_l}\right) \tag{6.3}$$

where γ is the one-dimensional second moment of G,

$$\gamma = \int_{-\infty}^{\infty} x^2\, dx \int_{-\infty}^{\infty} G(x, y, z)\, dy\, dz. \tag{6.4}$$

Finally, an expansion of $\bar{u}_i\bar{u}_j(x')$ including $O(|\mathbf{x} - \mathbf{x}'|)^2$ terms, gives

$$\frac{\partial(\overline{\bar{u}_i\bar{u}_j})}{\partial x_j} \simeq \frac{\partial}{\partial x_j}\left(\bar{u}_i\bar{u}_j + \frac{\gamma}{2}\frac{\partial^2}{\partial x_l\, \partial x_l}(\bar{u}_i\bar{u}_j)\right). \tag{6.5}$$

Using the methods of the previous section, one can show that the approximation (6.3) would yield twice the $\varepsilon_{\mathrm{RS}}$ determined for curve A, while (6.4) gives the same value of $\varepsilon_{\mathrm{RS}}$.

The SGS term $\partial\tau_{ij}/\partial x_j$ must produce the remaining dissipation and probably can be modeled by the eddy viscosity model of (3.11) and (3.12). The value of the eddy coefficient in (3.12) will be somewhat smaller than that calculated by Lilly (1967) ($c = 0.17$). Numerical experiments will probably be required to obtain a satisfactory value.

Similar considerations apply to the simulation of large-scale fluctuations of a passive scalar.

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